QUESTION BANK

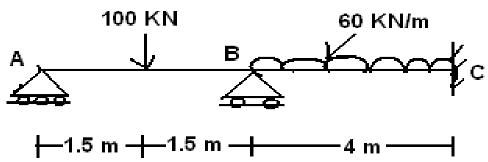
DEPARTMENT: CIVIL **SEMESTER: VI**

STRUCTURAL ANALYSIS-II

UNIT 1- FLEXIBILITY METHOD

PART - B (16 marks)

1. **Analyse the continuous beam shown in figure using force method. (AUC Apr/May 2011)**



Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (1 + 1 + 3) - 3 = 2

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

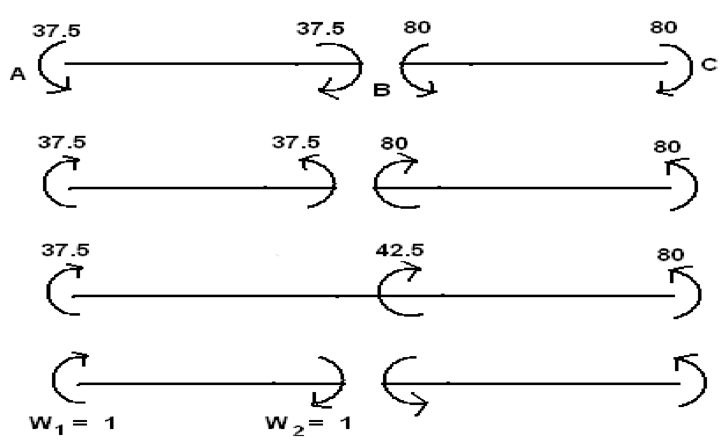
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{100 \text{ x } 3}{8} = -37.5 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{100 \text{ x } 3}{8} = 37.5 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{60 \text{ x } 4^2}{12} = -80 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell^2}{12} = \frac{60 \text{ x } 4^2}{12} = 80 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$$B = B_W \; B_X$$

$$B_W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

$$F_X = B_x^T \; F \; B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 2.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.502 & 0.253 \\ 0.253 & 0.879 \end{bmatrix}$$

$$F_W = B_x^T \, F \, B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 0.5 & -1 \\ 0 & 0 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \, F_W \, W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.502 & 0.253 \\ 0.253 & 0.879 \end{bmatrix}\begin{bmatrix} 0.5 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 37.5 \\ 42.5 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.251 & -0.502 \\ 0.127 & -0.253 \end{bmatrix}\begin{bmatrix} 37.5 \\ 42.5 \end{bmatrix}$$

$$= -\begin{bmatrix} -11.923 \\ -\ 5.99 \end{bmatrix}$$

$$X = \begin{bmatrix} 11.923 \\ 5.99 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 37.5 \\ 42.5 \\ 11.923 \\ 5.99 \end{bmatrix}$$

$$P = \begin{bmatrix} 37.5 \\ 30.58 \\ 11.923 \\ 5.99 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -37.5 \\ 37.5 \\ -80 \\ 80 \end{bmatrix} + \begin{bmatrix} 37.5 \\ 30.58 \\ 11.923 \\ 5.99 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 68.08 \\ -68.08 \\ 95.99 \end{bmatrix}$$

2. Analyse the portal frame ABCD shown in figure using force method. (AUC Apr/May 2011)

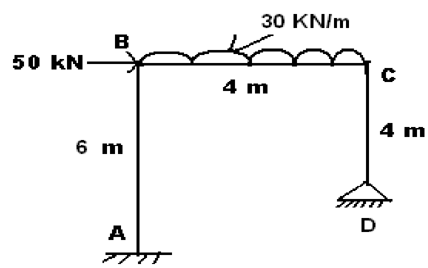


Fig.

Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (3 + 2) - 3 = 2

Release at B and C by apply hinge.

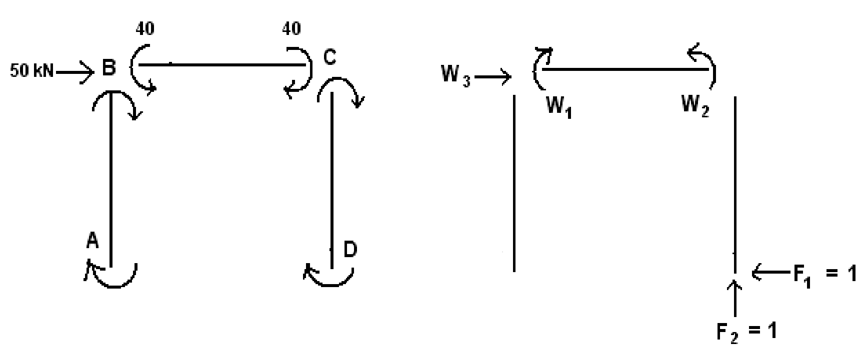
Apply a unit force at B joint.

Step 2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{30 \text{ x } 4^2}{12} = -40 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell^2}{12} = \frac{30 \text{ x } 4^2}{12} = 40 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$$B = B_W \; B_X$$

$$B_W = \begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -6 & -2 & 4 \\ 0 & 0 & 0 & 4 & -4 \\ 1 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

$$F_X = B_x^T \; F \; B_x$$

$$= \frac{1}{EI}\begin{bmatrix} -2 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -8 & 10 & -8 & 8 & -5.32 & 2.68 \\ 12 & -12 & 5.32 & -2.68 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI}\begin{bmatrix} 141.28 & -104 \\ -104 & 117.28 \end{bmatrix}$$

$$F_x^{-1} = EI\begin{bmatrix} 0.0204 & 0.0181 \\ 0.0181 & 0.0246 \end{bmatrix}$$

$$F_W = B_x^T \; F \; B_w$$

$$= \frac{1}{EI}\begin{bmatrix} -2 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -8 & 10 & -8 & 8 & -5.32 & 2.68 \\ 12 & -12 & 5.32 & -2.68 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} -8 & -5.32 & 48 \\ 5.32 & 0 & -72 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} F_W W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.0204 & 0.0181 \\ 0.0181 & 0.0246 \end{bmatrix}\begin{bmatrix} -8 & -5.32 & 48 \\ 5.32 & 0 & -72 \end{bmatrix}\begin{bmatrix} 40 \\ -40 \\ 50 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.0669 & 0.1085 & 0.3240 \\ 0.0139 & 0.0963 & 0.9024 \end{bmatrix}\begin{bmatrix} 40 \\ -40 \\ 50 \end{bmatrix}$$

$$= -\begin{bmatrix} -14.536 \\ -41.824 \end{bmatrix}$$

$$X = \begin{bmatrix} 14.536 \\ 41.824 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 & -2 & 4 \\ 0 & 0 & 0 & 4 & -4 \\ 1 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 40 \\ -40 \\ 50 \\ 14.536 \\ 41.824 \end{bmatrix}$$

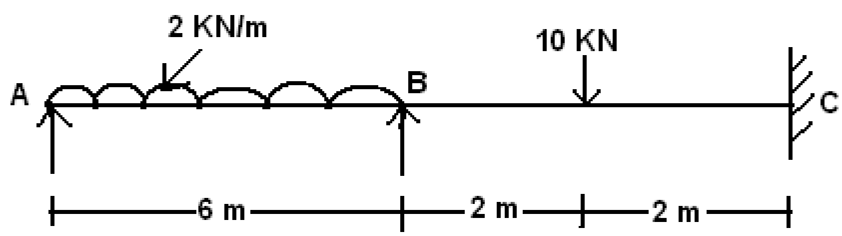
$$P = \begin{bmatrix} -161.776 \\ -109.152 \\ 149.152 \\ 58.144 \\ -98.144 \\ 0 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -40 \\ 40 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -161.776 \\ -109.152 \\ 149.152 \\ 58.144 \\ -98.144 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -161.776 \\ -109.152 \\ 109.152 \\ 98.144 \\ -98.144 \\ 0 \end{bmatrix}$$

3. Analyse the continuous beam ABC shown in figure by flexibility matrix method and sketch the bending moment diagram. (AUC Nov/Dec 2011).



EI = Constant

Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (1 + 1 + 3) - 3 = 2

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

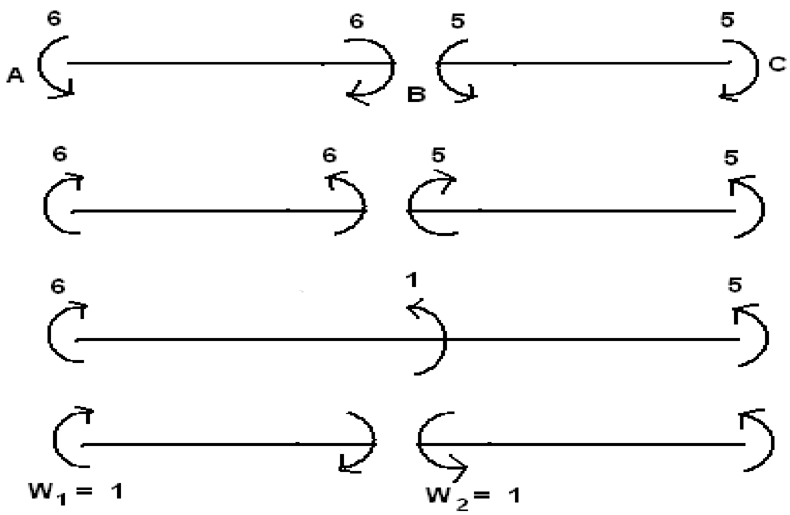
$$M_{FAB} = -\frac{w\ell^2}{12} = -\frac{2 \times 6^2}{12} = -6 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell^2}{12} = \frac{2 \times 6^2}{12} = 6 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{10 \times 4}{8} = -5 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{10 \times 4}{8} = 5 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$$B = B_W \, B_X$$

$$B_W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

$$F_X = B_x^T \, F \, B_x$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 1 & -2 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_x = \frac{1}{EI}\begin{bmatrix} 3.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$$

$$F_x^{-1} = EI\begin{bmatrix} 0.334 & 0.168 \\ 0.168 & 0.837 \end{bmatrix}$$

$$F_W = B_x^T \; F \; B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 1 & -2 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 1 & 1.33 \\ 0 & -0.67 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \; F_W \; W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.334 & 0.168 \\ 0.168 & 0.837 \end{bmatrix}\begin{bmatrix} 1 & 1.33 \\ 0 & -0.67 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.334 & 0.3316 \\ 0.168 & -0.337 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$= -\begin{bmatrix}1.672\\1.345\end{bmatrix}$$

$$X = \begin{bmatrix}-1.672\\-1.345\end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix}W\\X\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & -1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}6\\-1\\-1.672\\-1.345\end{bmatrix}$$

$$P = \begin{bmatrix}6\\1.672\\-2.672\\-1.345\end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix}-6\\6\\-5\\5\end{bmatrix} + \begin{bmatrix}6\\1.672\\-2.672\\-1.345\end{bmatrix}$$

$$M = \begin{bmatrix}0\\7.672\\-7.672\\3.655\end{bmatrix}$$

4. Analyse the portal frame ABCD shown in figure by flexibility matrix method and sketch the bending moment diagram. **(AUC Nov/Dec 2011)**

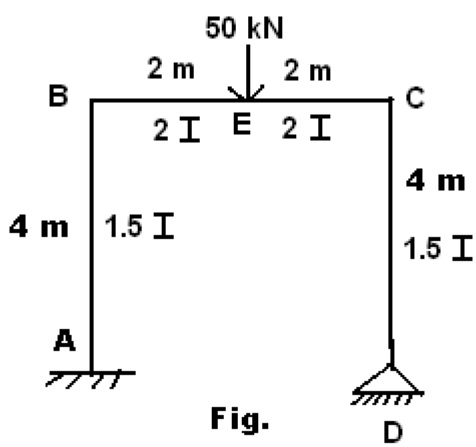


Fig.

Solution:

Step 1: Static Indeterminacy :

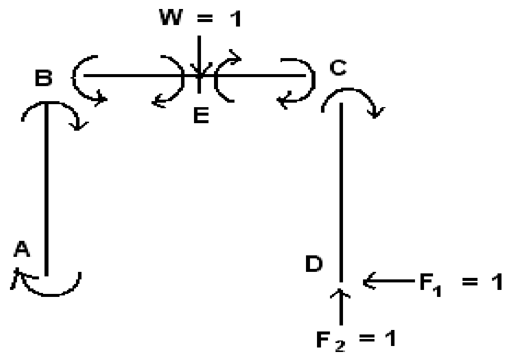
Degree of redundancy = (3 + 2) - 3 = 2

Release at D by apply horizontal and vertical supports.

Step 2: Fixed End Moment :

$M_{FAB} = M_{FBA} = M_{FBC} = M_{FBC} = M_{FCD} = M_{FDC} = 0$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$B = B_W \; B_X$

$$B_W = \begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 4 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI}\begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & -0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & -0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix}$$

$$F_X = B_x^T \, F \, B_x$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & 4 & -4 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & -2 & 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & -0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & -0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -1.76 & 3.56 & -2 & 2 & -2 & 2 & -3.56 & 1.76 \\ 5.32 & -5.32 & 1.66 & -1.34 & 0.66 & -0.34 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI}\begin{bmatrix} 60.48 & -37.28 \\ -37.28 & 53.2 \end{bmatrix}$$

$$F_x^{-1} = EI\begin{bmatrix} 0.0291 & 0.0203 \\ 0.0203 & 0.033 \end{bmatrix}$$

$$F_W = B_x^T \; F \; B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & 4 & -4 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & -2 & 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & -0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & -0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} -1.76 & 3.56 & -2 & 2 & -2 & 2 & -3.56 & 1.76 \\ 5.32 & -5.32 & 1.66 & -1.34 & 0.66 & -0.34 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 14.64 \\ -24.60 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} F_W W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.0291 & 0.0203 \\ 0.0203 & 0.033 \end{bmatrix}\begin{bmatrix} 14.64 \\ -24.60 \end{bmatrix} 50$$

$$= -\begin{bmatrix} -0.0734 \\ -0.5146 \end{bmatrix} 50$$

$$= -\begin{bmatrix} -3.67 \\ -25.73 \end{bmatrix}$$

$$X = \begin{bmatrix} 3.67 \\ 25.73 \end{bmatrix}$$

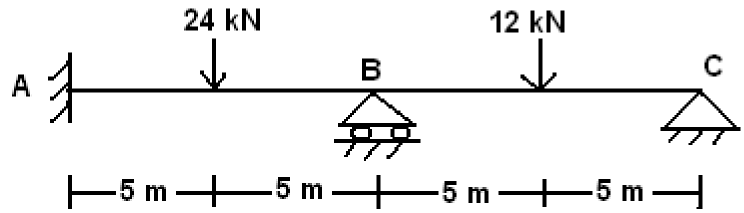
Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} -2 & 0 & 4 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 50 \\ 3.67 \\ 25.73 \end{bmatrix}$$

$$P = \begin{bmatrix} 2.92 \\ 11.76 \\ -11.76 \\ -36.78 \\ 36.78 \\ 14.68 \\ -14.68 \\ 0 \end{bmatrix}$$

The final moments also same, since there are no external forces acting on the members.

5. **Analyse the continuous beam ABC shown in figure by flexibility matrix method and sketch the bending moment diagram. (AUC May/June 2012)**



Solution:

Step1: Static Indeterminacy:

Degree of redundancy = (3 + 1 + 1) - 3 = 2

Release at A and B by apply hinge.

Step 2: Fixed End Moment:

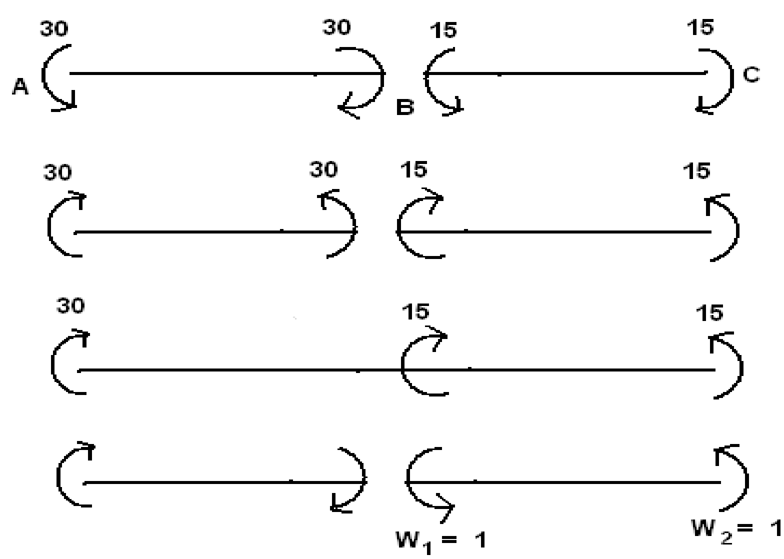
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{24 \text{ x } 10}{8} = -30 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{24 \text{ x } 10}{8} = 30 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{12 \text{ x } 10}{8} = -15 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell}{8} = \frac{12 \text{ x } 10}{8} = 15 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B) :

$$B = B_W \ B_X$$

$$B_W = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F) :

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix}$$

$$F_X = B_x^T \ F \ B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 3.33 & 1.67 \\ 1.67 & 6.66 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.3435 & -0.086 \\ -0.086 & 0.1717 \end{bmatrix}$$

$$F_W = B_x^T \, F \, B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \, F_W \; W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.3435 & -0.086 \\ -0.086 & 0.1717 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix}\begin{bmatrix} -15 \\ -15 \end{bmatrix}$$

$$= -\begin{bmatrix} -0.286 & 0.144 \\ 0.144 & -0.286 \end{bmatrix}\begin{bmatrix} -15 \\ -15 \end{bmatrix}$$

$$= -\begin{bmatrix} 2.13 \\ -4.29 \end{bmatrix}$$

$$X = \begin{bmatrix} -2.13 \\ 4.29 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -15 \\ -15 \\ -2.13 \\ 4.29 \end{bmatrix}$$

$$P = \begin{bmatrix} -\;2.13 \\ -\;4.29 \\ -10.71 \\ -15 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -30 \\ 30 \\ -15 \\ 15 \end{bmatrix} + \begin{bmatrix} -\ 2.13 \\ -\ 4.29 \\ -10.71 \\ -15 \end{bmatrix}$$

$$M = \begin{bmatrix} -32.13 \\ 25.71 \\ -25.71 \\ 0 \end{bmatrix}$$

6. **A cantilever of length 15 m is subjected to a single concentrated load of 50 kN at the middle of the span. Find the deflection at the free end using flexibility matrix method. EI is uniform throughout. (AUC May/June 2013)**

Solution:

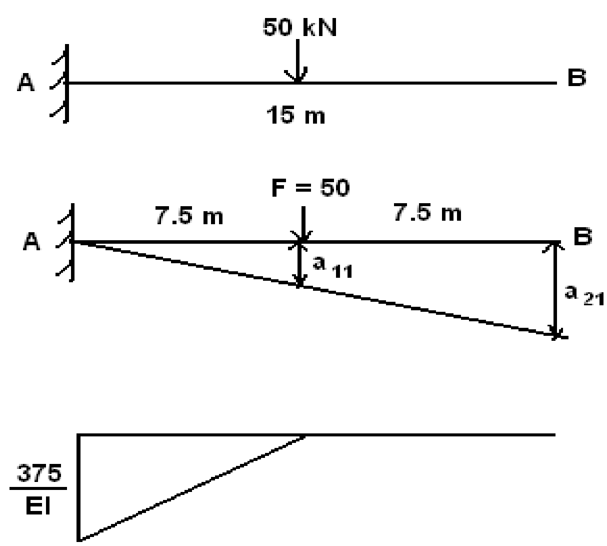
Step1: Static Indeterminacy :

Degree of redundancy = 3 - 3 = 0

It is static determinate structures.

Step 2: Deflection at B :

Apply a unit force at given load.



The deflection is calculated by $\frac{M}{EI}$.

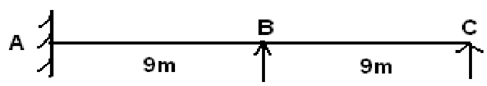
$$\text{Deflection at } a_{21} = \left(\frac{1}{2} \times 7.5 \times \frac{375}{EI}\right) \times \left(\frac{2 \times 7.5}{3} + 7.5\right)$$

$$\text{Deflection at B} = \frac{17578.125}{EI}$$

Hint: To find the deflection, we use $\frac{M}{EI}$ diagram.

7. A two span continuous beam ABC is fixed at A and hinged at support B and C. Span AB = BC = 9m. Set up flexibility influence coefficient matrix assuming vertical reaction at B and C as redundant. (AUC May/June 2013)

Solution:



Step 1: Static Indeterminacy:

Degree of redundancy = (3 + 1 + 1) - 3 = 2

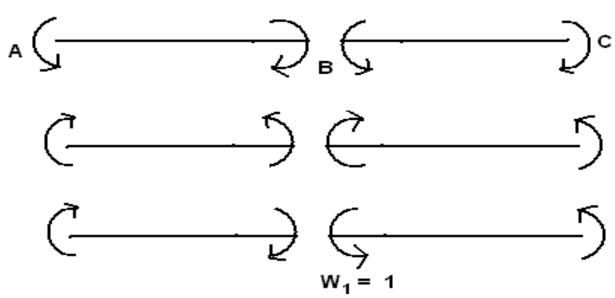
Release at A and B by apply hinge.

Step 2: Fixed End Moment:

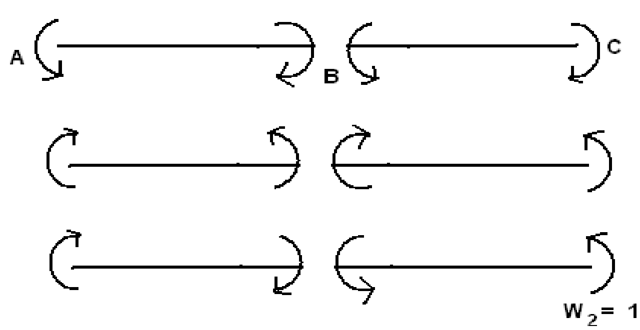
$M_{FAB} = M_{FBA} = M_{FBC} = M_{FBC} = 0$

Step 3: Equivalent Joint Load:

Case (i):



Case (ii):



Step 3 : Flexibility Influence Co-efficient Matrix (B):

For case (i):

$$B = B_W \; B_X$$

$$B_W = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

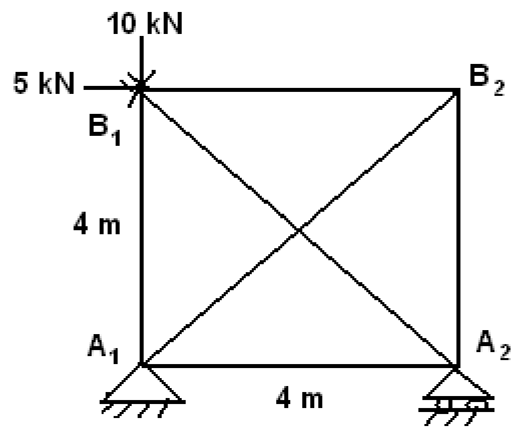
For case (ii):

$$B = B_W \; B_X$$

$$B_W = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

8. A Statically indeterminate frame shown in figure carries a load of 80 kN. Analyse the frame by matrix flexibility method. A and E are same for all members. (AUC May/June 2012)



Solution:

Step 1: Static Indeterminacy:

Degree of redundancy = Internal Indeterminate – External Indeterminate

= [m – (2j – 3)] – (r – R)

= [6 – (8 – 3)] – (3 - 3)

= 1

Step 2: Member forces:

Take member AD as a redundant.

$\tan\theta = \frac{3}{4} = 0.75$; $\sin\theta = 0.6$; $\cos\theta = 0.8$;

<u>ΣV = 0</u>

$V_A = 1$

<u>ΣM = 0</u>

$H_A = 1.333$ and $H_B = 1.333$

<u>At joint D:</u>

$F_{DC} = 1$ (compression) = -1

<u>At joint C:</u>

<u>ΣV = 0</u>

$F_{CA} \sin\theta = 1$

$F_{CA} = 1.667$; $F_{CB} = 1.333$

<u>At joint B:</u>

$F_{BA} = 0$; $F_{BC} = 1.333$

Analyse by method of joints and find the member forces.

Step 3: Flexibility Co-efficient Matrix:

$$B = B_W \; B_X$$

$$B_W = \begin{bmatrix} 0 \\ -1.333 \\ -1 \\ 1.667 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \\ -1.25 \\ -1.25 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0.75 \\ -1.333 & 1 \\ -1 & 0.75 \\ 1.667 & -1.25 \\ 0 & -1.25 \\ 0 & 1 \end{bmatrix}$$

Step 4: Flexibility matrix (F):

$$F = \frac{1}{AE}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$F_X = B_x^T \, F \, B_x$$

$$= \begin{matrix} 0.75 & 1 & 0.75 & -1.25 & -1.25 & 1 \end{matrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \\ -1.25 \\ -1.25 \\ 1 \end{bmatrix}$$

$$F_x = \frac{27}{AE}$$

$$F_x^{-1} = \frac{AE}{27}$$

$$F_W = B_x^T \, F \, B_w$$

$$= \begin{matrix} 0.75 & 1 & 0.75 & -1.25 & -1.25 & 1 \end{matrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -1.333 \\ -1 \\ 1.667 \\ 0 \\ 0 \end{bmatrix}$$

$$F_W = -\frac{7.30}{AE}$$

Step 5 : Displacement matrix (X) :

$$X = -F_x^{-1} F_W W$$

$$= \left[\frac{-AE}{27}\right]\left[\frac{-7.30}{AE}\right] 80$$

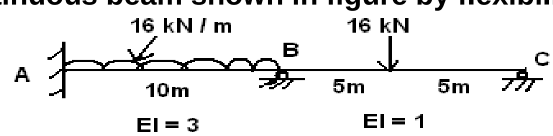
$$X = 21.63 \text{ kN}$$

Step 6 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0.75 \\ -1.333 & 1 \\ -1 & 0.75 \\ 1.667 & -1.25 \\ 0 & -1.25 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 80 \\ 21.63 \end{bmatrix}$$

$$\text{Final forces, } P = \begin{bmatrix} 16.22 \\ -84.77 \\ -63.78 \\ 105.76 \\ -27.04 \\ 21.63 \end{bmatrix}$$

9. Analyse the continuous beam shown in figure by flexibility method.



Solution:

Step 1: Static Indeterminacy :

Degree of redundancy = (3 + 1 + 1) - 3 = 2

Release at A and B by apply hinge.

Step 2: Fixed End Moment :

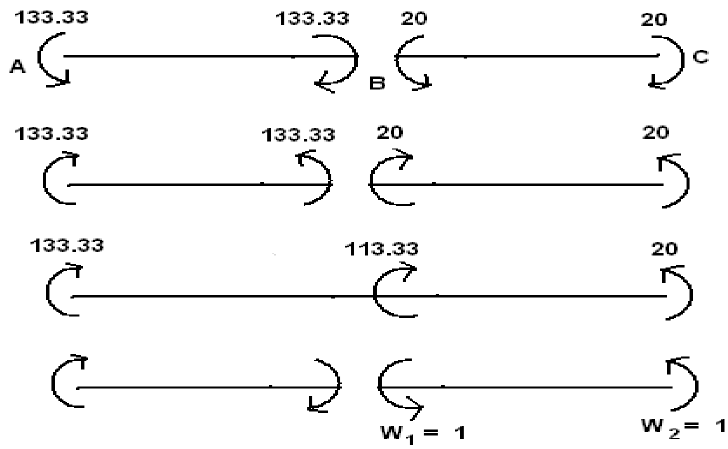
$$M_{FAB} = -\frac{w\ell^2}{12} = -\frac{16 \times 10^2}{12} = -133.33 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell^2}{12} = \frac{16 \times 10^2}{12} = 133.33 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{16 \times 10}{8} = -20 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{16 \times 10}{8} = 20 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$$B = B_W \; B_X$$

$$B_W = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ -0.56 & 1.11 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix}$$

$$F_X = B_x^T \; F \; B_x$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ -0.56 & 1.11 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ 0.56 & -1.11 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \begin{bmatrix} 1.11 & 0.56 \\ 0.56 & 4.44 \end{bmatrix}$$

$$F_x^{-1} = \begin{bmatrix} 0.962 & -0.121 \\ -0.121 & 0.241 \end{bmatrix}$$

$$F_W = B_x^T \ F \ B_w$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ -0.56 & 1.11 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ 0.56 & -1.11 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_W = \begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \ F_W \ W$$

$$= -\begin{bmatrix} 0.962 & -0.121 \\ -0.121 & 0.241 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} -113.33 \\ -20 \end{bmatrix}$$

$$= -\begin{bmatrix} 41.62 \\ -82.90 \end{bmatrix}$$

$$X = \begin{bmatrix} -41.62 \\ 82.90 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -113.33 \\ -20 \\ -41.62 \\ 82.90 \end{bmatrix}$$

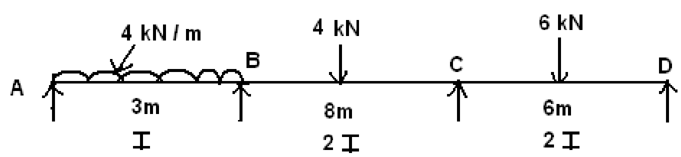
$$P = \begin{bmatrix} -41.62 \\ -82.90 \\ -30.43 \\ -20 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -133.33 \\ 133.33 \\ -20 \\ 20 \end{bmatrix} + \begin{bmatrix} -41.62 \\ -82.90 \\ -30.43 \\ -20 \end{bmatrix}$$

$$M = \begin{bmatrix} -174.95 \\ 50.43 \\ -50.43 \\ 0 \end{bmatrix}$$

10. Analyse the continuous beam shown in figure by flexibility method.



Solution:

Step1: Static Indeterminacy :

Degree of redundancy = (1 + 1 + 1 + 1) - 2 = 2

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

$$M_{FAB} = -\frac{w\ell^2}{12} = -\frac{4 \times 3^2}{12} = -3 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell^2}{12} = \frac{4 \times 3^2}{12} = 3 \text{ kNm}$$

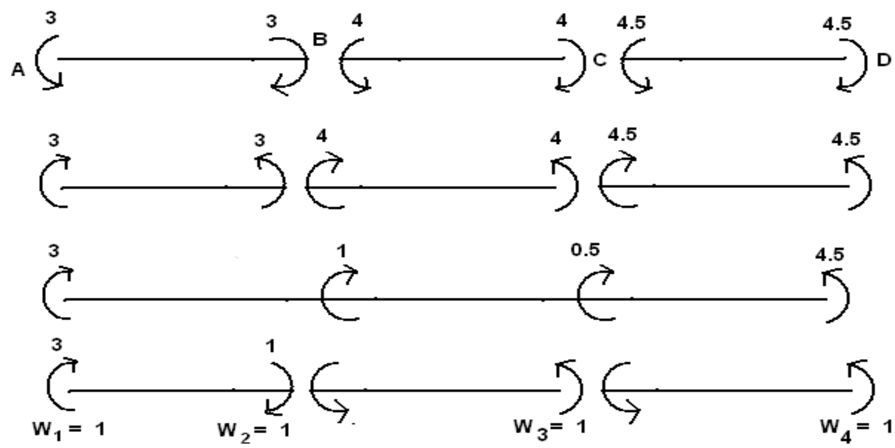
$$M_{FBC} = -\frac{w\ell}{8} = -\frac{4 \text{ x } 8}{8} = -4 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{4 \text{ x } 8}{8} = 4 \text{ kNm}$$

$$M_{FCD} = -\frac{w\ell}{8} = -\frac{6 \text{ x } 6}{8} = -4.5 \text{ kNm}$$

$$M_{FDC} = \frac{w\ell}{8} = \frac{6 \text{ x } 6}{8} = 4.5 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co – efficient matrix (B):

$$B = B_W \; B_X$$

$$B_W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B_X = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI}\begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix}$$

$$F_X = B_x^T \; F \; B_x$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & 0.67 & -1.33 & 1 & -0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI}\begin{bmatrix} 2.33 & 0.67 \\ 0.67 & 2.33 \end{bmatrix}$$

$$F_x^{-1} = EI\begin{bmatrix} 0.468 & -0.135 \\ -0.135 & 0.468 \end{bmatrix}$$

$$F_W = B_x^T \; F \; B_w$$

$$= \frac{1}{EI}\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & 0.67 & -1.33 & 1 & -0.5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_W = \frac{1}{EI}\begin{bmatrix} 0.5 & -1 & -0.67 & 0 \\ 0 & 0 & -1.33 & -0.5 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$X = -F_x^{-1} \, F_W \;\; W$$

$$= -\frac{EI}{EI}\begin{bmatrix} 0.468 & -0.135 \\ -0.135 & 0.468 \end{bmatrix}\begin{bmatrix} 0.5 & -1 & -0.67 & 0 \\ 0 & 0 & -1.33 & -0.5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0.5 \\ -4.5 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.234 & -0.468 & -0.134 & 0.068 \\ -0.068 & 0.135 & -0.599 & -0.234 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0.5 \\ -4.5 \end{bmatrix}$$

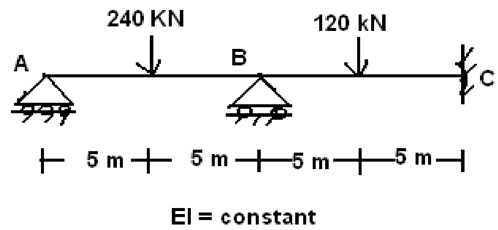
$$= -\begin{bmatrix} -0.139 \\ 0.685 \end{bmatrix}$$

$$X = \begin{bmatrix} 0.139 \\ -0.685 \end{bmatrix}$$

PART - B (16 marks)

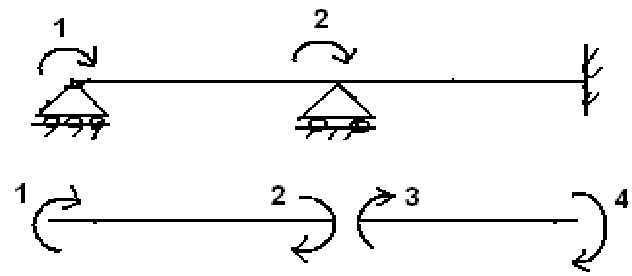
1. Analyse the continuous beam shown in figure using displacement method.

(AUC Apr/May 2011)



Solution:

Step1: Assign coordinates :



Step2: Fixed End Moment :

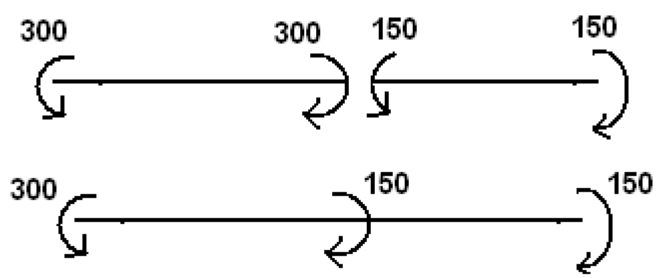
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{240 \text{ x } 10}{8} = -300 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{240 \text{ x } 10}{8} = 300 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{120 \text{ x } 10}{8} = -150 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{120 \text{ x } 10}{8} = 150 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -300 \\ 150 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 2.86 & -0.71 \\ -0.71 & 1.43 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 2.86 & -0.71 \\ -0.71 & 1.43 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -300 \\ 150 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\ A\ \Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \\ 0 & 0.4 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

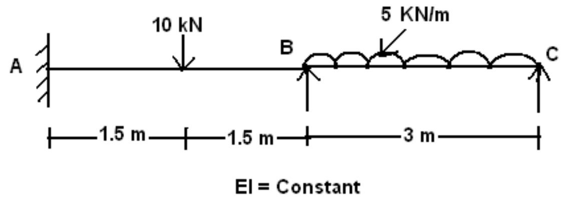
$$P = \begin{bmatrix} 300 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -300 \\ 300 \\ -150 \\ 150 \end{bmatrix} + \begin{bmatrix} 300 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

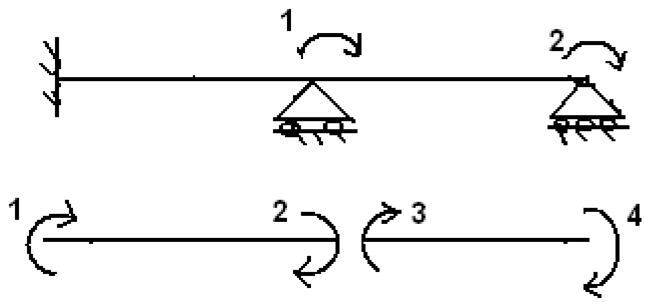
$$M = \begin{bmatrix} 0 \\ 321.9 \\ -321 \\ 64.5 \end{bmatrix}$$

2. **Analyse the continuous beam ABC shown in figure by stiffness method and also draw the shear force diagram. (AUC Nov/Dec 2011).**



Solution:

Step1: Assign coordinates :



Step2: Fixed End Moment :

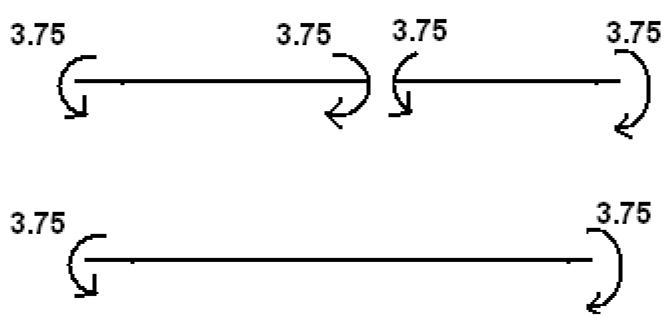
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \text{ x } 3}{8} = -3.75 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \text{ x } 3}{8} = 3.75 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{5 \text{ x } 3^2}{12} = -3.75 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{5 \text{ x } 3^2}{12} = 3.75 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} 0 \\ 3.75 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}$$

Step 6 :System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.67 & 1.33 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 2.66 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.431 & -0.217 \\ -0.217 & 0.861 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.431 & -0.217 \\ -0.217 & 0.861 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 3.75 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\, A\, \Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

$$= \begin{bmatrix} 0.67 & 0 \\ 1.33 & 0 \\ 1.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

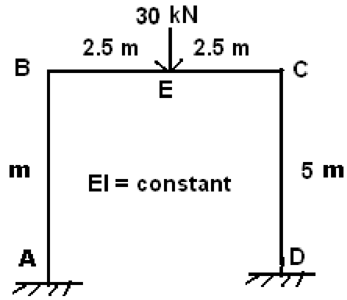
$$P = \begin{bmatrix} 0.545 \\ 1.082 \\ -1.081 \\ -3.75 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3.75 \\ 3.75 \\ -3.75 \\ 3.75 \end{bmatrix} + \begin{bmatrix} 0.545 \\ 1.082 \\ -1.081 \\ -3.75 \end{bmatrix}$$

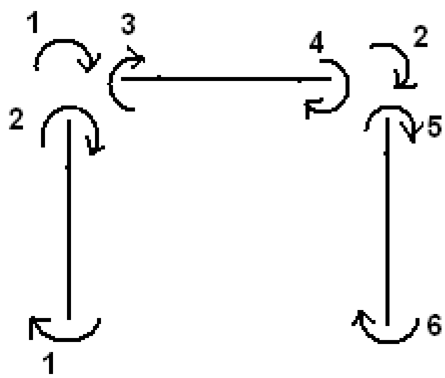
$$M = \begin{bmatrix} -3.205 \\ 4.832 \\ -4.832 \\ 0 \end{bmatrix}$$

3. Analyse the portal frame ABCD shown in figure by stiffness method and also draw the bending moment diagram. (AUC Nov/Dec 2011)



Solution:

Step1: Assign coordinates :



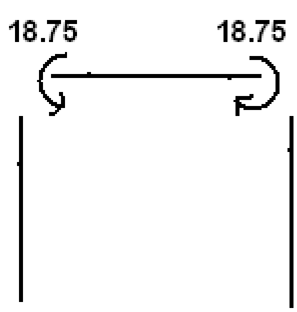
Step2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{30 \text{ x } 5}{8} = -18.75 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell}{8} = \frac{30 \text{ x } 5}{8} = 18.75 \text{ kNm}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -18.75 \\ 18.75 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.4 & 0.8 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 1.6 & 0.4 \\ 0.4 & 1.6 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.67 & -0.17 \\ -0.17 & 0.67 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.67 & -0.17 \\ -0.17 & 0.67 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -18.75 \\ 18.75 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

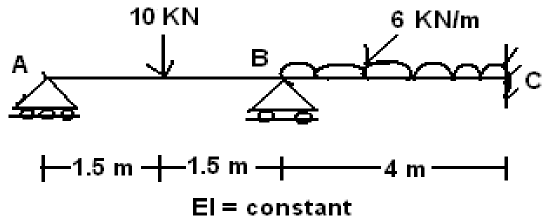
$$= \begin{bmatrix} 0.4 & 0 \\ 0.8 & 0 \\ 0.8 & 0.4 \\ 0.4 & 0.8 \\ 0 & 0.8 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

$$P = \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix}$$

Step 9 : Final Moments (M):

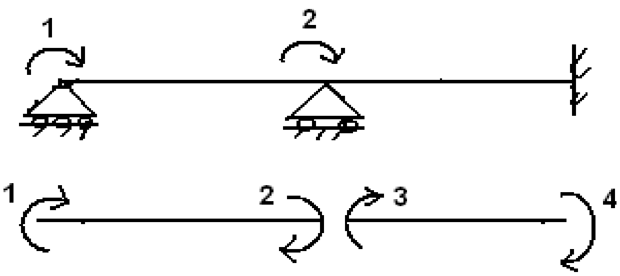
$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -18.75 \\ 18.75 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix} = \begin{bmatrix} 6.3 \\ 12.6 \\ -12.5 \\ 12.5 \\ -12.6 \\ -\ 6.3 \end{bmatrix}$$

4. Analyse the continuous beam ABC shown in figure by stiffness method and also sketch the bending moment diagram. (AUC May/June 2012)



Solution:

Step1: Assign coordinates :



Step2: Fixed End Moment :

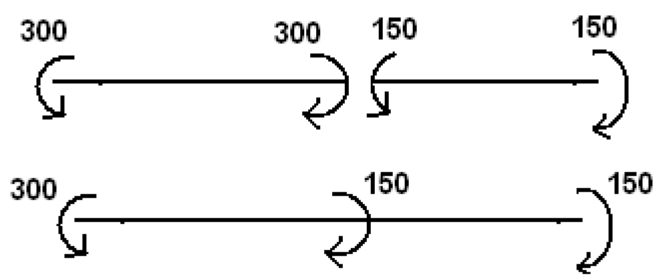
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \text{ x } 3}{8} = -3.75 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \text{ x } 3}{8} = 3.75 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{6 \text{ x } 4^2}{12} = -8 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{6 \text{ x } 4^2}{12} = 8 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -3.75 \\ -4.25 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 1.33 & 0.67 \\ 0.67 & 2.33 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.879 & -0.253 \\ -0.253 & 0.502 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.879 & -0.253 \\ -0.253 & 0.502 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -3.75 \\ -4.25 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 2.221 \\ 1.185 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\ A\ \Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2.221 \\ 1.185 \end{bmatrix}$$

$$= \begin{bmatrix} 1.33 & 0.67 \\ 0.67 & 1.33 \\ 0 & 1 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 2.221 \\ 1.185 \end{bmatrix}$$

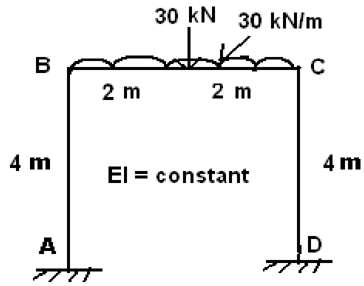
$$P = \begin{bmatrix} 3.75 \\ 3.06 \\ 1.185 \\ 0.59 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3.75 \\ 3.75 \\ -8 \\ 8 \end{bmatrix} + \begin{bmatrix} 3.75 \\ 3.06 \\ 1.185 \\ 0.59 \end{bmatrix}$$

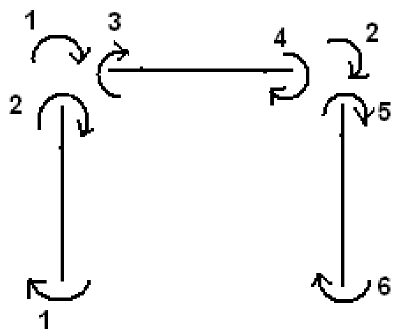
$$M = \begin{bmatrix} 0 \\ 6.81 \\ -6.81 \\ 8.59 \end{bmatrix}$$

5. Analyse the portal frame ABCD shown in figure by stiffness method and also sketch the bending moment diagram. (AUC May/June 2012)



Solution:

Step1: Assign coordinates :



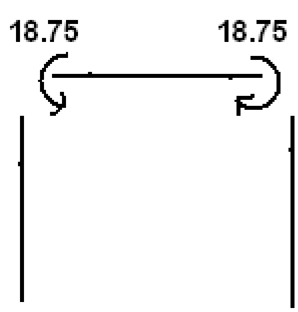
Step2: Fixed End Moment :

$$M_{FBC} = -\left[\frac{w\ell}{8} + \frac{w\ell^2}{12}\right] = -\left[\frac{30 \text{ x } 4}{8} + \frac{30 \text{ x } 4^2}{12}\right] = -55 \text{ kNm}$$

$$M_{FCB} = \left[\frac{w\ell}{8} + \frac{w\ell^2}{12}\right] = \left[\frac{30 \text{ x } 4}{8} + \frac{30 \text{ x } 4^2}{12}\right] = 55 \text{ kNm}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -55 \\ 55 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.5 & 1 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.53 & -0.13 \\ -0.13 & 0.53 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.53 & -0.13 \\ -0.13 & 0.53 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -55 \\ 55 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \\ 1 & 0.5 \\ 0.5 & 1 \\ 0 & 1 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

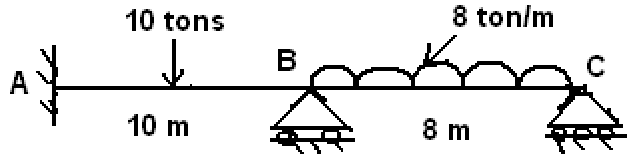
$$P = \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix}$$

Step 9 : Final Moments (M):

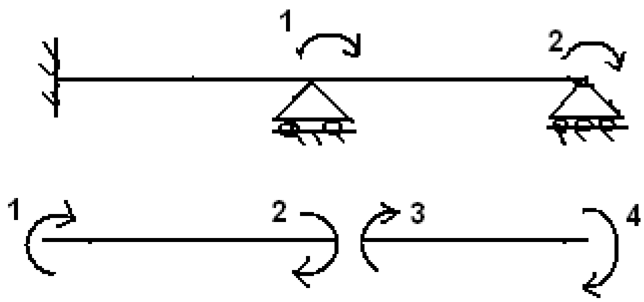
$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -55 \\ 55 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix} = \begin{bmatrix} 18.15 \\ 36.3 \\ -36.3 \\ 36.45 \\ -36.3 \\ -18.15 \end{bmatrix}$$

6. A two span continuous beam ABC is fixed at A and simply supported over the supports B and C. AB = 10 m and BC = 8 m. moment of inertia is constant throughout. A single central concentrated load of 10 tons acts on AB and a uniformly distributed load of 8 ton/m acts over BC. Analyse the beam by stiffness matrix method. (AUC May/June 2013)

Solution:



Step1: Assign coordinates :



Step2: Fixed End Moment :

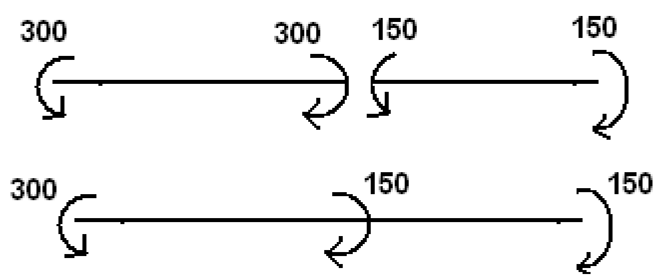
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \text{ x } 10}{8} = -12.5 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \text{ x } 10}{8} = 12.5 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{8 \text{ x } 8^2}{12} = -42.67 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell^2}{12} = \frac{8 \text{ x } 8^2}{12} = 42.67 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -30.17 \\ 42.67 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.2 & 0.4 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 0.9 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 1.29 & -0.65 \\ -0.65 & 2.32 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 1.29 & -0.65 \\ -0.65 & 2.32 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -30.17 \\ 42.67 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\ A\ \Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0 \\ 0.4 & 0 \\ 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

$$P = \begin{bmatrix} 13.33 \\ 26.66 \\ 3.68 \\ -42.64 \end{bmatrix}$$

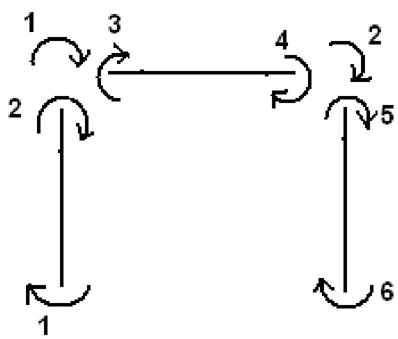
Step 9: Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -12.5 \\ 12.5 \\ -42.67 \\ 42.67 \end{bmatrix} + \begin{bmatrix} 13.33 \\ 26.66 \\ 3.68 \\ -42.64 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.83 \\ 39.16 \\ -39 \\ 0 \end{bmatrix}$$

7. A portal frame ABCD with supports A and D are fixed at same level carries a uniformly distributed load of 8 tons/m on the span AB. Span AB = BC = CD = 9 m. EI is constant throughout. Analyse the frame by stiffness matrix method. (AUC May/June 2013)
Solution:

Step1: Assign coordinates :



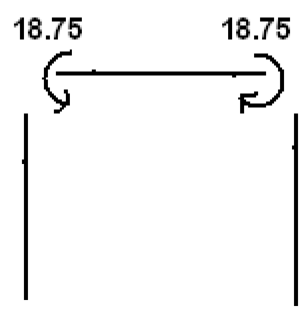
Step2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{8 \times 9^2}{12} = -54 \text{ ton.m}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{8 \times 9^2}{12} = 54 \text{ ton.m}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -54 \\ 54 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.22 & 0.44 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0.44 & 0.22 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 0.88 & 0.22 \\ 0.22 & 0.88 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 1.212 & -0.303 \\ -0.303 & 1.212 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 1.212 & -0.303 \\ -0.303 & 1.212 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -54 \\ 54 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

$$= \begin{bmatrix} 0.22 & 0 \\ 0.44 & 0 \\ 0.44 & 0.22 \\ 0.22 & 0.44 \\ 0 & 0.44 \\ 0 & 0.22 \end{bmatrix}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

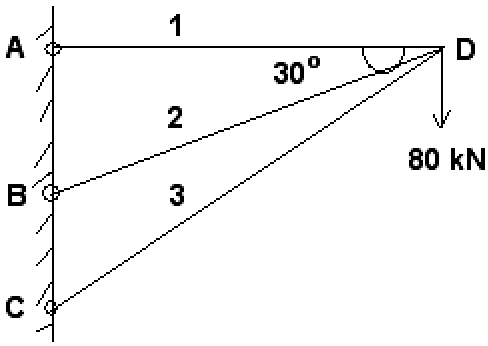
$$P = \begin{bmatrix} 18 \\ 36 \\ 18 \\ -18 \\ -36 \\ -18 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -54 \\ 54 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 36 \\ 18 \\ -18 \\ -36 \\ -18 \end{bmatrix} = \begin{bmatrix} 18 \\ 36 \\ -36 \\ 36 \\ -36 \\ -18 \end{bmatrix}$$

8. Using matrix stiffness method, analyze the truss for the member forces in the truss loaded as shown in figure. AE and L are tabulated below for all the three members.

(AUC Apr/May 2011)

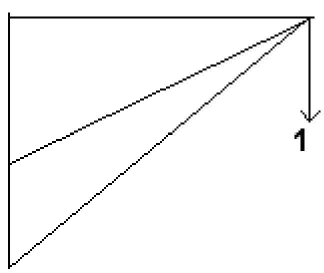


Member	AE	L
AD	400	400
BD	461.9	461.9
CD	800	800

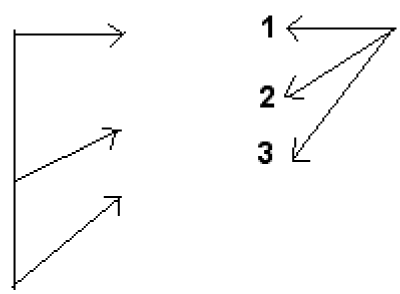
Solution:

Step 1: Assign coordinates:

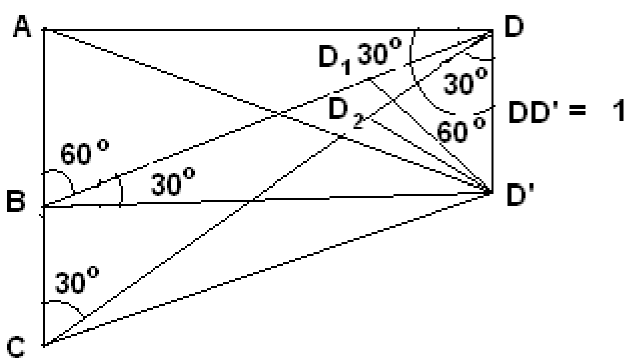
i) Global coordinates:



ii) Local coordinates:



Step 2: Displacement diagram:



Step 3: Formation of [A] matrix:

Apply unit displacement in DD'.

Displacement along 1, AD = 0

Displacement along 2 and 3,

$DD_1 = \cos 60^o = 0.5$ and $DD_2 = \cos 30^o = 0.866$

$$A = \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

Step 4: Stiffness matrix (K):

$$K = \frac{AE}{L}\begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 5: System stiffness matrix (J):

$$J = A^T K A$$

$$= \begin{bmatrix} 0 & -0.5 & -0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.5 & -0.866 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

$$J = 1$$

$$J^{-1} = 1$$

Step 6: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= 1 \text{ x } 80 = 80$$

Step 7: Element forces (P):

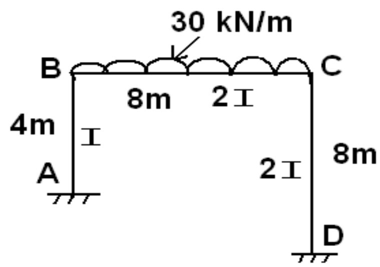
$$P = K A \Delta$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix} 80$$

$$= \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix} 80$$

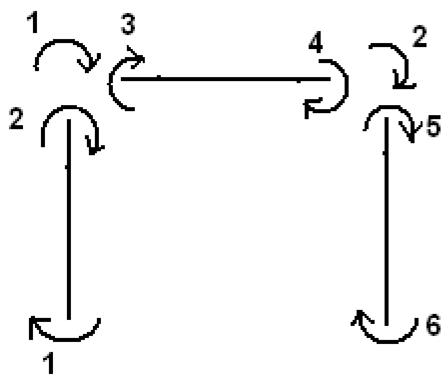
$$\text{Final forces, } P = \begin{bmatrix} 0 \\ -40 \\ -69.28 \end{bmatrix}$$

9. Analyse the frame shown in figure by matrix stiffness method.



Solution:

Step1: Assign coordinates :



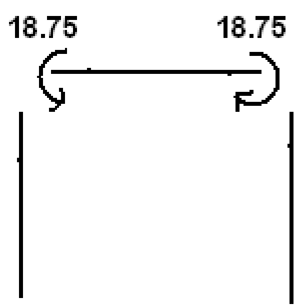
Step 2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{30 \times 8^2}{12} = -160 \text{ kN.m}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{30 \times 8^2}{12} = 160 \text{ kN.m}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} 0 \\ -160 \\ 160 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.25 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.125 & 0 & 1 \\ -0.125 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -0.25 & -0.25 & 0 & 0 & -0.125 & -0.125 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6 : System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} -0.25 & -0.25 & 0 & 0 & -0.125 & -0.125 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} -0.375 & -0.375 & 0 & 0 & -0.187 & -0.187 \\ 0.5 & 1 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 0.234 & -0.375 & -0.187 \\ -0.375 & 2 & 0.5 \\ -0.187 & 0.5 & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 6.29 & 1.10 & 0.31 \\ 1.10 & 0.73 & -0.08 \\ 0.31 & -0.08 & 0.55 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 6.29 & 1.10 & 0.31 \\ 1.10 & 0.73 & -0.08 \\ 0.31 & -0.08 & 0.55 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -160 \\ 160 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 126.4 \\ 129.6 \\ -100.8 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

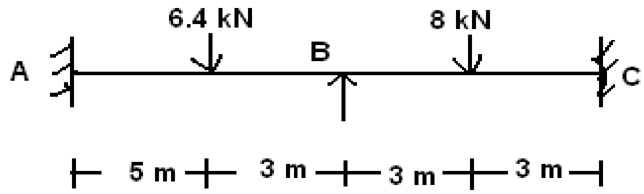
$$= \frac{EI}{EI}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix}\begin{bmatrix} 126.4 \\ 129.6 \\ -100.8 \end{bmatrix}$$

$$P = \begin{bmatrix} -0.375 & 0.5 & 0 \\ -0.375 & 1 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 1 \\ -0.187 & 0 & 1 \\ -0.187 & 0 & 0.5 \end{bmatrix}\begin{bmatrix} 126.4 \\ 129.6 \\ -100.8 \end{bmatrix} = \begin{bmatrix} 17.4 \\ 82.2 \\ 79.2 \\ 36 \\ -124.44 \\ -74.04 \end{bmatrix}$$

Step 9: Final Moments (M):

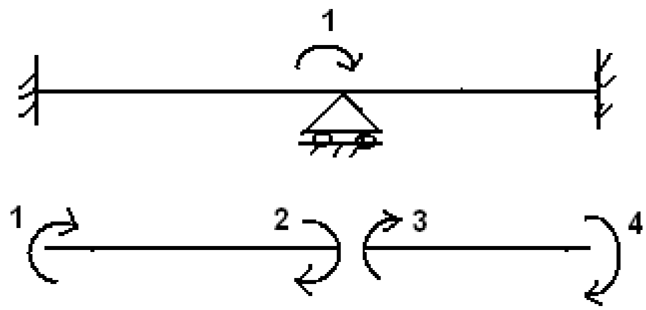
$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -160 \\ 160 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 17.4 \\ 82.2 \\ 79.2 \\ 36 \\ -124.44 \\ -74.04 \end{bmatrix} = \begin{bmatrix} 17.4 \\ 82.2 \\ -81 \\ 124 \\ -124.44 \\ -74.04 \end{bmatrix}$$

10. Analyse the continuous beam shown in figure using displacement method.



Solution:

Step1: Assign coordinates :



Step 2: Fixed End Moment :

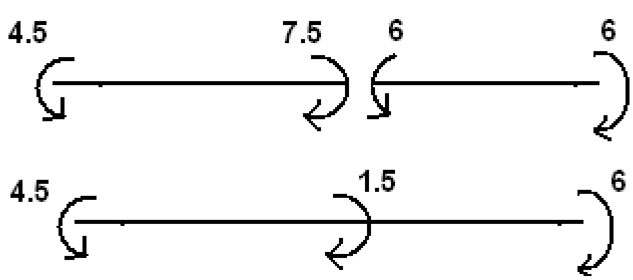
$$M_{FAB} = -\frac{wab^2}{\ell^2} = -\frac{6.4 \text{ x } 5 \text{ x } 3^2}{8^2} = -4.5 \text{ kNm}$$

$$M_{FBA} = \frac{wa^2b}{\ell^2} = -\frac{6.4 \text{ x } 5^2 \text{ x } 3}{8^2} = 7.5 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{8 \text{ x } 6}{8} = -6 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{8 \text{ x } 6}{8} = 6 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$W^O = 1.5$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI\begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.25 & 0.5 & 0.67 & 0.33 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$J = EI\ 1.17$$

$$J^{-1} = \frac{0.85}{EI}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{0.85}{EI} \quad 0 - 1.5$$

$$\Delta = -\frac{1.275}{EI}$$

Step 8 : Element forces (P):

$$P = K\ A\ \Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad -1.275$$

$$= \begin{bmatrix} 0.25 \\ 0.5 \\ 0.67 \\ 0.33 \end{bmatrix} \quad -1.275$$

$$P = \begin{bmatrix} -0.319 \\ -0.638 \\ -0.854 \\ -0.421 \end{bmatrix}$$

Step 9 : Final Moments (M):

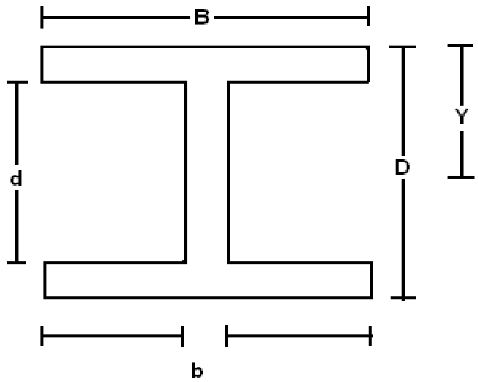
$$M = \mu + P = \begin{bmatrix} -4.5 \\ 7.5 \\ -6 \\ 6 \end{bmatrix} + \begin{bmatrix} -0.319 \\ -0.638 \\ -0.854 \\ -0.421 \end{bmatrix}$$

$$M = \begin{bmatrix} -4.82 \\ 6.86 \\ -6.85 \\ 5.58 \end{bmatrix}$$

PART - B (16 marks)

1. Derive the shape factor for I section and circular section. (AUC Apr/May 2011)

I section:



Shape factor, $S = \frac{Z_p}{Z} = \frac{\text{Plastic modulus}}{\text{Elastic modulus}}$

Elastic modulus (Z):

$$Z = \frac{I}{Y}$$

$$I = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$Y = \frac{D}{2}$$

$$Z = \frac{\left(\frac{BD^3}{12} - \frac{bd^3}{12}\right)}{\left(\frac{D}{2}\right)} = \left(\frac{BD^3}{12} - \frac{bd^3}{12}\right) x \ \frac{2}{D}$$

$$Z = \frac{BD^3 - bd^3}{6D}$$

Plastic modulus (Z_p):

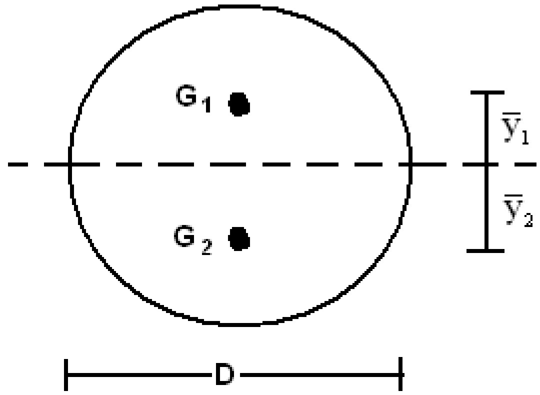
$$Z_p = \frac{A}{2}(\bar{y}_1 + \bar{y}_2)$$

$$A = 2(b_1 d_1) + b_2 d_2$$

$$\bar{y}_1 = \bar{y}_2 = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$S = \frac{Z_p}{Z} = \frac{\frac{A}{2}(\bar{y}_1 + \bar{y}_2)}{\left(\frac{BD^3 - bd^3}{6D}\right)}$$

Circular Section:



Shape factor, S = $\dfrac{Z_p}{Z} = \dfrac{\text{Plastic modulus}}{\text{Elastic modulus}}$

Elastic modulus (Z):

$$Z = \frac{I}{y} = \frac{\left(\dfrac{\pi D^4}{64}\right)}{\dfrac{D}{2}}$$

$$Z = \frac{\pi D^3}{32}$$

Plastic modulus (Z_p):

$$Z_p = \frac{A}{2} \quad \bar{y}_1 + \bar{y}_2$$

$$A = \frac{\pi D^2}{4}$$

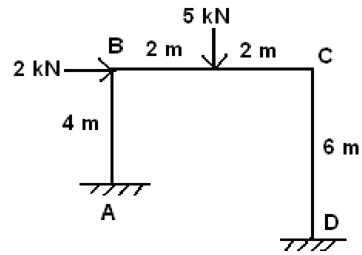
$$\bar{y}_1 = \bar{y}_2 = \frac{4r}{3\pi} = \frac{2D}{3\pi}$$

$$Z_P = \frac{\pi D^2}{4\times 2}\left(\frac{2D}{3\pi} + \frac{2D}{3\pi}\right) = \frac{\pi D^2}{8} \times \frac{4D}{3\pi} = \frac{D^3}{6}$$

$$S = \frac{Z_P}{Z} = \frac{\left(\dfrac{D^3}{6}\right)}{\left(\dfrac{\pi D^3}{32}\right)} = \frac{D^3}{6} \times \frac{32}{\pi D^3} = \frac{32}{6\pi}$$

$$S = 1.697$$

2. **Find the fully plastic moment required for the frame shown in figure, if all the members have same value of M_P.** **(AUC Apr/May 2011)**



Solution:

Step 1: Degree of indeterminacy:

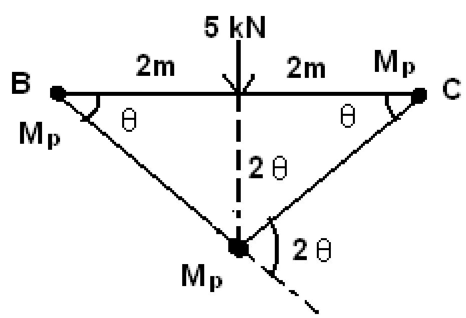
Degree of indeterminacy = (No. of closed loops x 3) – No. of releases

= (1 x 3) – 0 = 3

No. of possible plastic hinges = 5

No. of independent mechanisms = 5 – 3 = 2

Step 2: Beam Mechanism:



$$EWD = 5(2\theta) = 10\theta$$

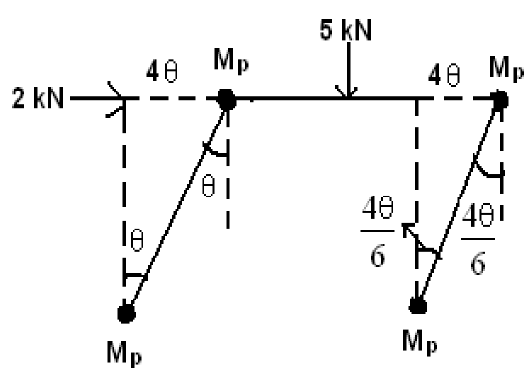
$$IWD = M_p\theta + 2M_p\theta + M_p\theta = 4M_p\theta$$

$$EWD = IWD$$

$$10\theta = 4M_p\theta$$

$$M_p = 2.5\,kN.m$$

Step 3: Sway Mechanism:



$$EWD = (2 \times 4\theta) = 8\theta$$

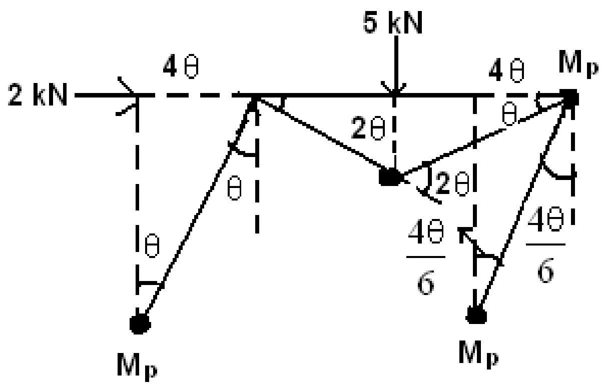
$$IWD = M_p\theta + M_p\theta + M_p\left(\frac{4\theta}{6}\right) + M_p\left(\frac{4\theta}{6}\right) = 3.33M_p\theta$$

$$EWD = IWD$$

$$8\theta = 3.33M_p\theta$$

$$M_p = 2.4\,kN.m$$

Step 4: Combined Mechanism:



$$EWD = (2 \times 4\theta) + (5 \times 2\theta) = 18\theta$$

$$IWD = M_p\theta + M_p(2\theta) + M_p\left(\theta + \frac{4\theta}{6}\right) + M_p\left(\frac{4\theta}{6}\right) = 5.33M_p\theta$$

$$EWD = IWD$$

$$18\theta = 5.33M_p\theta$$

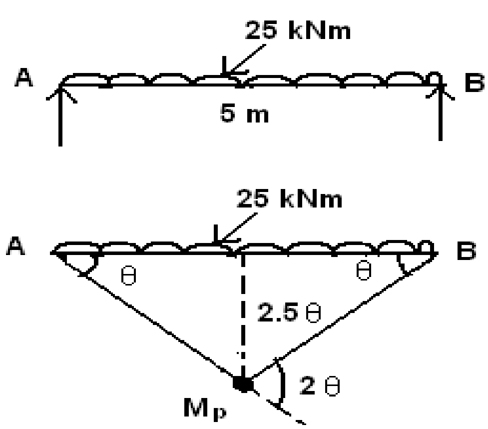
$$M_p = 3.38\,kN.m$$

The fully plastic moment, M_P = 3.38 kNm.

3. A simply supported beam of span 5 m is to be designed for an udl of 25 kN/m. Design a suitable I section using plastic theory, assuming yield stress in steel as $f_y = 250\ N/mm^2$.

(AUC Nov/Dec 2011)

Solution:



$$IWD = 0 + M_P(2\theta) + 0 = 2M_P\theta$$

EWD = Load intensity X area of triangle under the load

$$= 25 \text{ x } \left(\frac{1}{2} \text{ x } 5 \text{ x } 2.5\theta\right)$$

$$= 156.25\theta$$

$$IWD = EWD$$

$$2M_P\theta = 156.25\theta$$

$$M_P = 78.125 \text{ kNm}$$

W.K.T.,

$$M_P = \sigma_y \text{ x } Z_P$$

$$Z_P = \frac{M_P}{\sigma_y} = \frac{78.125 \text{ x } 10^6}{250} = 3.12 \text{ x } 10^5 \text{ mm}^3$$

Assuming the shape factor for I-section as 1.15

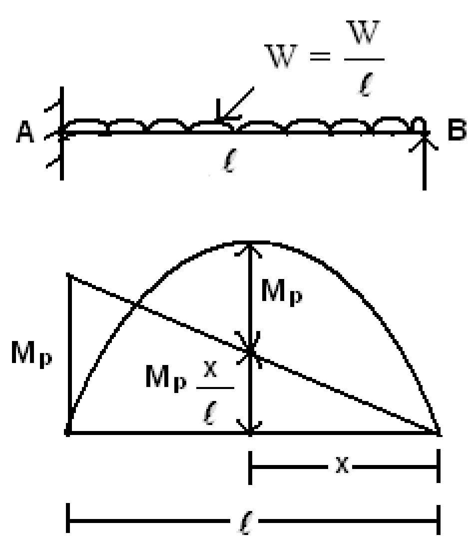
$$S = \frac{Z_P}{Z}$$

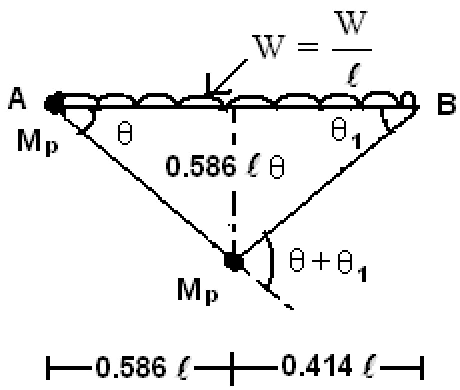
$$Z = \frac{Z_P}{S} = \frac{3.12 \text{ x } 10^5}{1.15} = 271.74 \text{ x } 10^3 \text{ mm}^3.$$

Adopt ISLB 250 @ 279 N / m (from steel table)

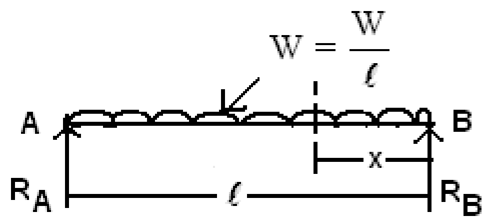
4. Analyse a propped cantilever of length 'L' and subjected to udl of w/m length for the entire span and find the collapse load. (AUC Nov/Dec 2011)

Solution:





Consider the moment at A as redundant and that it reaches M_P. the second hinge will form where the net positive BM is maximum.



$$\sum V = 0$$

$$R_A + R_B = W_C$$

$$R_A = R_B = \frac{W_C}{2}$$

$$M_x = \frac{W_C X}{2} - \frac{W_C X^2}{2\ell}$$

$$M_P + \frac{M_P X}{\ell} = \frac{W_C X}{2} - \frac{W_C X^2}{2\ell}$$

$$M_P\left(1 + \frac{X}{\ell}\right) = \frac{W_C X}{2}\left(1 - \frac{X}{\ell}\right)$$

$$M_P\left(\frac{\ell + X}{\ell}\right) = \frac{W_C X}{2}\left(\frac{\ell - X}{\ell}\right)$$

$$M_P = \frac{W_C X}{2}\left(\frac{\ell - X}{\ell + X}\right) = \frac{W_C}{2}\left(\frac{\ell X - X^2}{\ell + X}\right)$$

For M_P to be maximum, $\frac{dM_P}{dx} = 0$

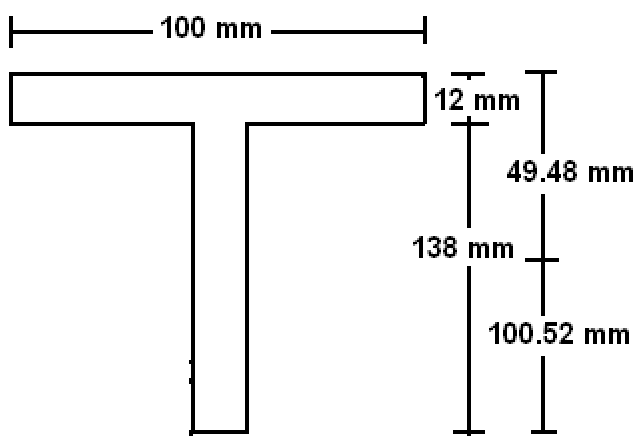
$$\frac{dM_P}{dx} = \frac{W_C}{2}\left[\frac{(\ell + x)(\ell - 2x) - (\ell x - x^2)(1)}{(\ell + x)^2}\right] = 0$$

$$(\ell + x)(\ell - 2x) - (\ell x - x^2) = 0$$
$$\ell^2 - 2\ell x + x\ell - 2x^2 - \ell x + x^2 = 0$$
$$\ell^2 - 2\ell x - x^2 = 0$$
$$x^2 + 2\ell x - \ell^2 = 0$$
$$x = \frac{-2\ell \pm \sqrt{8\ell^2}}{2}$$
$$x = 0.414\ell$$

Mechanism:

$$0.586\,\ell\theta = 0.414\,\ell\theta_1$$
$$\theta_1 = 1.4155\,\theta$$
$$\theta + \theta_1 = \theta + 1.4155\theta = 2.4155\,\theta$$
$$\text{EWD} = \frac{W_C}{\ell} \text{ x } \frac{1}{2} \text{ x } \ell \text{ x } 0.586\,\ell\theta = 0.293\,W_C\,\ell\,\theta$$
$$\text{IWD} = M_P\,\theta + M_P\,(2.4155\,\theta) + 0 = 3.4155\,M_P\theta$$
$$\text{EWD} = \text{IWD}$$
$$0.293\,W_C\,\ell\,\theta = 3.4155\,M_P\theta$$
$$W_C = \frac{11.66\,M_P}{\ell}$$

5. **Determine the shape factor of a T-section beam of flange dimension 100 x 12 mm and web dimension 138 x 12 mm thick. (AUC May/June 2012)**
   Solution:



Shape factor, $S = \frac{Z_P}{Z} = \frac{\text{Plastic modulus}}{\text{Elastic modulus}}$

i) Elastic modulus (Z_e) :

$$\bar{y}_t = \frac{(100 \times 12 \times 6) + (12 \times 138 \times 81)}{(100 \times 12) + (12 \times 138)} = 49.48\,\text{mm}$$

$$\bar{y}_b = 150 - 49.48 = 100.52\ \text{mm}$$

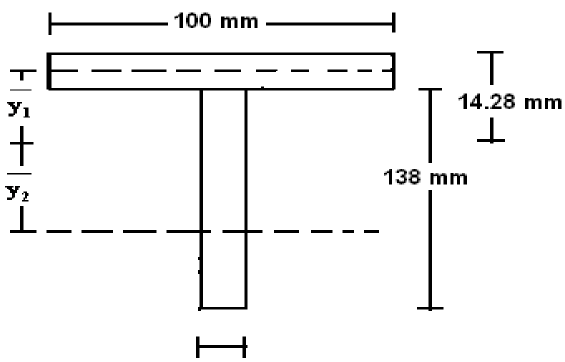
$$I_{xx} = \left[\frac{b_1 d_1^3}{12} + A_1 h_1^2\right] + \left[\frac{b_2 d_2^3}{12} + A_2 h_2^2\right]$$

$$= \left[\frac{100 \times 12^3}{12} + (100 \times 12 \times 43.48^2)\right] + \left[\frac{12 \times 138^3}{12} + (10 \times 138 \times 31.52^2)\right]$$

$$I_{xx} = 6.27 \times 10^6\ \text{mm}^4$$

$$Z_e = \frac{I}{y_{max}} = \frac{6.27 \times 10^6}{100.52} = 62375.65\ \text{mm}^3$$

ii) Plastic modulus :



Equal area axis,

$$\frac{A}{2} = \text{width of the flange X h}$$

$$\frac{2856}{2} = 100\,h$$

$$h = 14.28\ \text{mm (from top)}$$

$$\bar{y}_1 = \frac{(100 \times 12 \times (6 + 2.28)) + (12 \times 135.72 \times 67.86)}{(100 \times 12) + (12 \times 135.72)} = 42.58\,\text{mm}$$

$$\bar{y}_2 = \frac{107.42}{2} = 53.71\ \text{mm}$$

$$Z_p = \frac{A}{2}(\bar{y}_1 + \bar{y}_2) = \frac{2856}{2}(42.58 + 53.71)$$

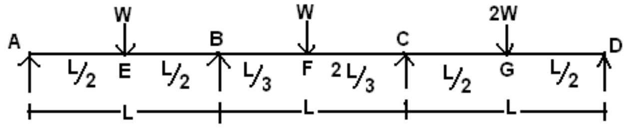
$$Z_p = 137502.12\ \text{mm}^3$$

Shape factor,

$$S = \frac{Z_P}{Z} = \frac{137502.12}{62375.65}$$

$$S = 2.20$$

6. Determine the collapse load 'W' for a three span continuous beam of constant plastic moment 'M_P' loaded as shown in figure. (AUC May/June 2012)



Solution:

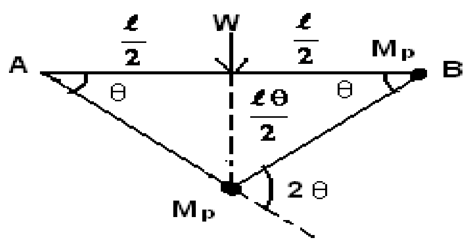
Step 1: Degree of indeterminacy:

Degree of indeterminacy = 4 – 2 = 2

No. of possible plastic hinges = 5

No. of independent mechanisms = 5 – 2 = 3

Step 2: Mechanism (1):



$$EWD = W \times \frac{\ell\theta}{2} = \frac{W\ell\theta}{2}$$

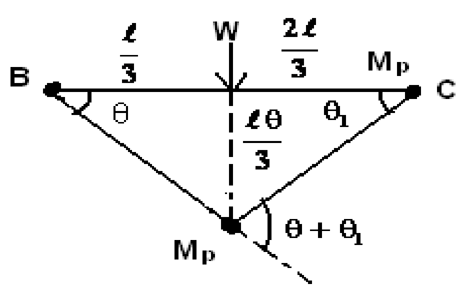
$$IWD = M_P(2\theta) + M_P\theta = 3M_P\theta$$

$$IWD = EWD$$

$$3M_P\theta = \frac{W\ell\theta}{2}$$

$$W_C = \frac{6M_P}{\ell}$$

Step 3: Mechanism (2):



$$\frac{\ell\theta}{3} = \frac{2\ell\theta_1}{3}$$

$$\theta_1 = \frac{\theta}{2}$$

$$\theta + \theta_1 = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$$

$$\text{EWD} = W \times \frac{\ell\theta}{3} = \frac{W\ell\theta}{3}$$

$$\text{IWD} = M_P\theta + M_P(\theta + \theta_1) + M_P\theta_1$$

$$= M_P\theta + \frac{3M_P\theta}{2} + \frac{M_P\theta}{2}$$

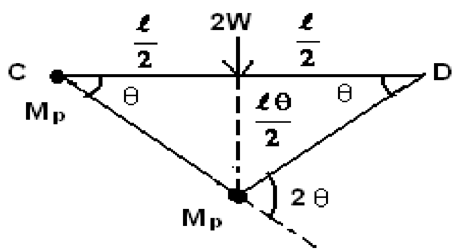
$$= 3M_P\theta$$

$$\text{IWD} = \text{EWD}$$

$$3M_P\theta = \frac{W\ell\theta}{3}$$

$$W_C = \frac{9M_P}{\ell}$$

Step 4: Mechanism (3):



$$\text{EWD} = 2W \times \frac{\ell\theta}{2} = W\ell\theta$$

$$\text{IWD} = M_P\theta + M_P(2\theta) = 3M_P\theta$$

$$\text{IWD} = \text{EWD}$$

$$3M_P\theta = W\ell\theta$$

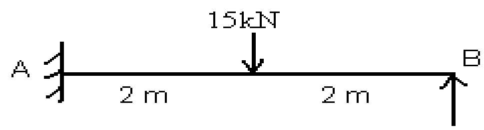
$$W_C = \frac{3M_P}{\ell}$$

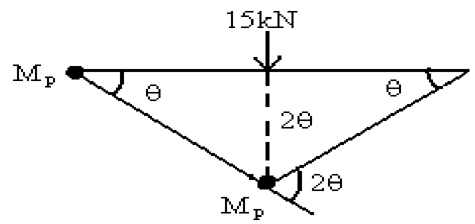
The collapse load $W_C = \frac{3M_P}{\ell}$ and the beam will fail.

7. **A uniform beam of span 4 m and fully plastic moment M_P is simply supported at one end and rigidly clamped at other end. A concentrated load of 15 kN may be applied anywhere within the span. Find the smallest value of M_P such that collapse would first occur when the load is in its most unfavourable position. (AUC May/June 2013)**

Solution:

i) When the load is at centre:

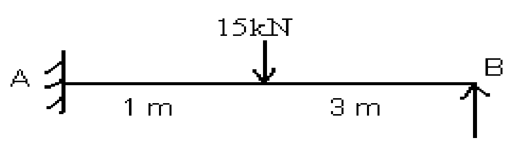


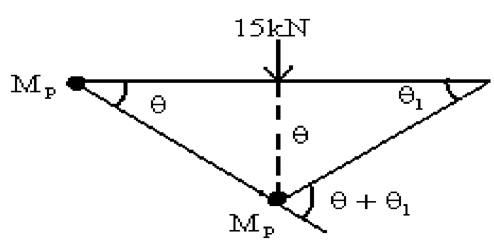


Degree of indeterminacy = 4 – 3 = 1
No. of possible plastic hinges = 2
No. of independent mechanisms = 2 – 1 = 1

$$EWD = 15\,(2\theta) = 30\,\theta$$
$$IWD = M_P\theta + M_P(2\theta) = 3M_P\theta$$
$$IWD = EWD$$
$$3M_P\theta = 30\,\theta$$
$$M_P = 10\text{ kNm}$$

ii) When the load is at unfavourable position:





$$1 \text{ x } \theta = 3 \text{ x } \theta_1$$
$$\theta_1 = \frac{\theta}{3}$$

$$EWD = 15\theta$$

$$IWD = M_P\theta + M_P(\theta + \theta_1) = M_P\theta + M_P\left(\theta + \frac{\theta}{3}\right)$$

$$= \frac{7}{3} M_P\theta$$

$$IWD = EWD$$

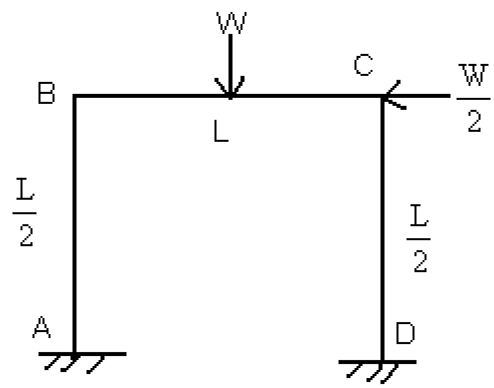
$$\frac{7}{3} M_P\theta = 15\theta$$

$$M_P = 6.43 \text{ kNm}$$

The smallest value of M_P is 6.43 kNm.

8. A rectangular portal frame of span L and $L/2$ is fixed to the ground at both ends and has a uniform section throughout with its fully plastic moment of resistance equal to M_y. It is loaded with a point load W at centre of span as well as a horizontal force $W/2$ at its top right corner. Calculate the value of W at collapse of the frame. (AUC May/June 2013)

Solution:



Step 1: Degree of indeterminacy:

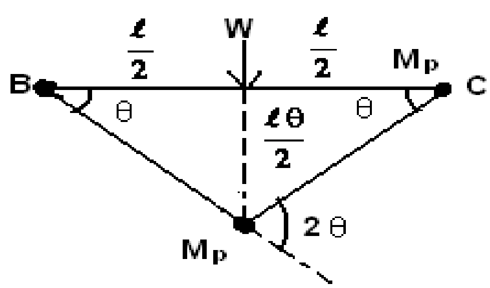
Degree of indeterminacy = (No. of closed loops x 3) – No. of releases

= (1 x 3) – 0 = 3

No. of possible plastic hinges = 5

No. of independent mechanisms = 5 – 3 = 2

Step 2: Beam Mechanism:



$$EWD = \frac{W\ell\theta}{2}$$

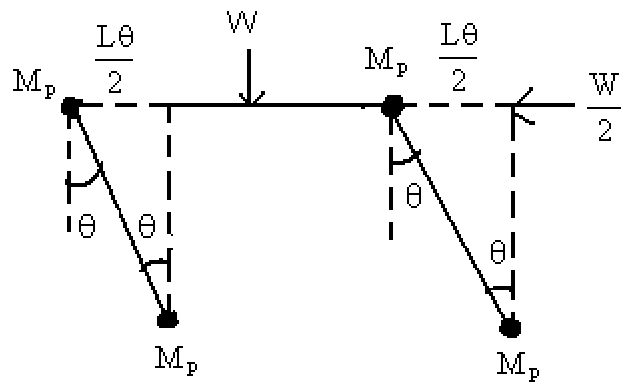
$$IWD = M_p\theta + M_p(2\theta) + M_p\theta = 4M_p\theta$$

$$EWD = IWD$$

$$\frac{W\ell\theta}{2} = 4M_p\theta$$

$$W_C = \frac{8M_p}{\ell}$$

Step 3: Sway Mechanism:



$$EWD = \frac{W\ell\theta}{4}$$

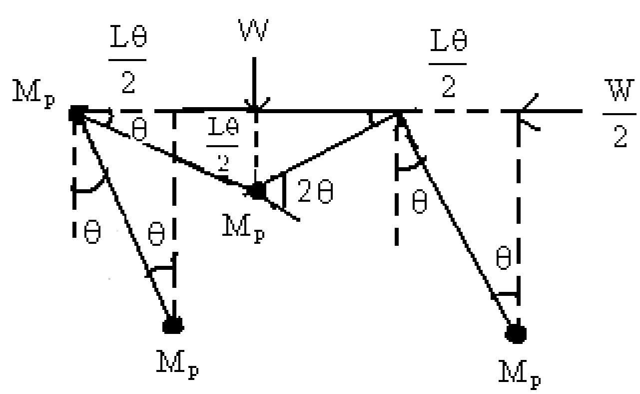
$$IWD = M_p\theta + M_p\theta + M_p\theta + M_p\theta = 4M_p\theta$$

$$EWD = IWD$$

$$\frac{W\ell\theta}{4} = 4M_p\theta$$

$$W_C = \frac{16\,M_p}{\ell}$$

Step 4: Combined Mechanism:



$$EWD = \left(\frac{W\ell\theta}{2}\right) + \left(\frac{W\ell\theta}{4}\right) = \frac{3W\ell\theta}{4}$$

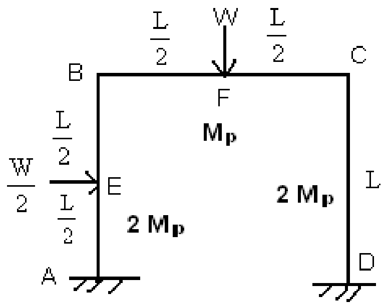
$$IWD = M_p\theta + M_p(2\theta) + M_p(2\theta) + M_p\theta = 6M_p\theta$$

$$EWD = IWD$$

$$\frac{3W\ell\theta}{4} = 6M_p\theta$$

$$W_C = \frac{8M_p}{\ell}$$

9. Find the collapse load for the frame shown in figure.



Solution:

Step 1: Degree of indeterminacy:

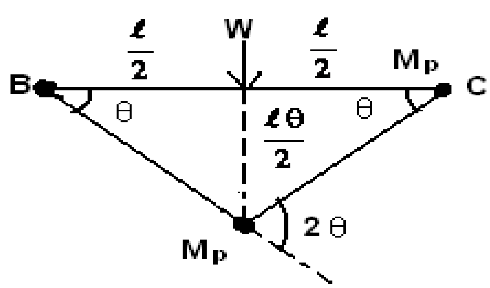
Degree of indeterminacy = (No. of closed loops x 3) – No. of releases

= (1 x 3) – 1 = 2

No. of possible plastic hinges = 5

No. of independent mechanisms = 5 – 2 = 3

Step 2: Beam Mechanism:



$$EWD = \frac{W\ell\theta}{2}$$

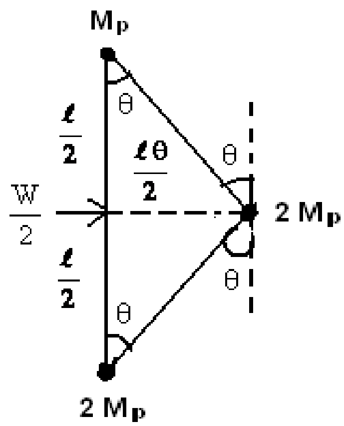
$$IWD = M_p\theta + M_p(2\theta) + M_p\theta = 4M_p\theta$$

$$EWD = IWD$$

$$\frac{W\ell\theta}{2} = 4M_p\theta$$

$$W_C = \frac{8M_p}{\ell}$$

Step 3: Column Mechanism:



$$EWD = \frac{W\ell\theta}{4}$$

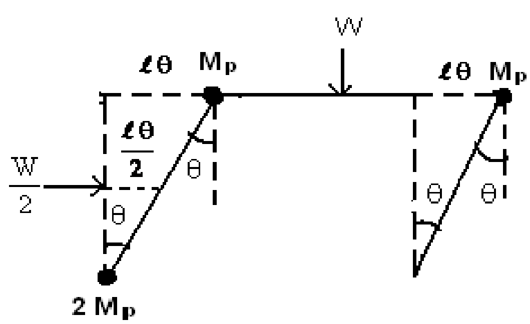
$$IWD = 2M_p\theta + 2M_p(2\theta) + M_p\theta = 7M_p\theta$$

$$EWD = IWD$$

$$\frac{W\ell\theta}{2} = 7M_p\theta$$

$$W_C = \frac{28M_p}{\ell}$$

Step 3: Sway Mechanism:



$$EWD = \frac{W\ell\theta}{4}$$

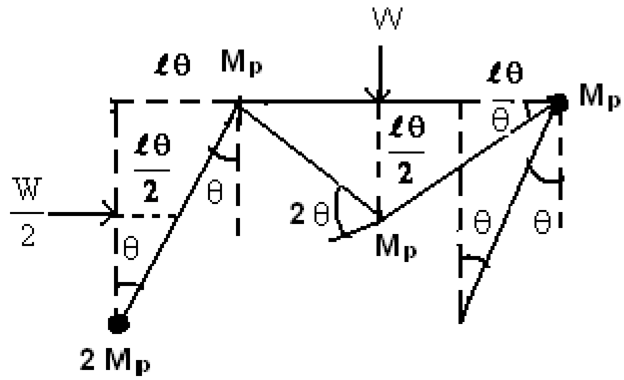
$$IWD = 2M_p\theta + M_p\theta + M_p\theta = 4M_p\theta$$

$$EWD = IWD$$

$$\frac{W\ell\theta}{4} = 4M_p\theta$$

$$W_C = \frac{16\,M_p}{\ell}$$

Step 4: Combined Mechanism:



$$EWD = \left(\frac{W\ell\theta}{4}\right) + \left(\frac{W\ell\theta}{2}\right) = \frac{3W\ell\theta}{4}$$

$$IWD = 2M_p\theta + M_p(2\theta) + M_p(2\theta) = 6M_p\theta$$

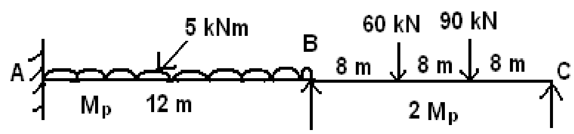
$$EWD = IWD$$

$$\frac{3W\ell\theta}{4} = 6M_p\theta$$

$$W_C = \frac{8\,M_p}{\ell}$$

Hence the collapse load, $W_C = \frac{8\,M_p}{\ell}$

10. A continuous beam ABC is loaded as shown in figure. Determine the required M_P if the load factor is 3.2.



Solution:

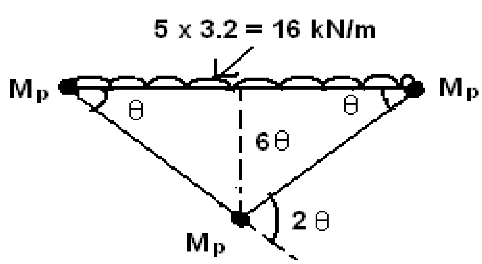
Step 1: Degree of indeterminacy:

Degree of indeterminacy = 5 – 3 = 2

No. of possible plastic hinges = 5

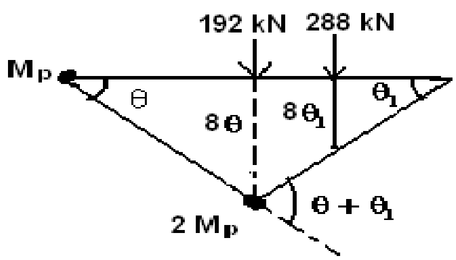
No. of independent mechanisms = 5 – 2 = 3

Step 2: Mechanism (1):



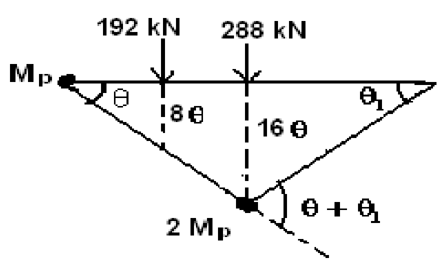
$$EWD = 16 \times \frac{1}{2} \times 12 \times 6\theta$$
$$= 576\,\theta$$
$$IWD = M_p\theta + M_p(2\theta) + M_p\theta$$
$$= 4M_p\theta$$
$$IWD = EWD$$
$$4M_p\theta = 576\,\theta$$
$$M_p = 144 \text{ kNm}$$

Step 3: Mechanism (2):



$$8\,\theta = 16\,\theta_1$$
$$\theta_1 = \frac{\theta}{2}$$
$$\theta + \theta_1 = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$$
$$EWD = (192 \times 8\theta) + (288 \times 4\theta) = 2688\,\theta$$
$$IWD = M_p\theta + 2M_p(\theta + \theta_1) = 4M_p\theta$$
$$IWD = EWD$$
$$4M_p\theta = 2688\,\theta$$
$$M_p = 672 \text{ kNm}$$

Step 4: Mechanism (3):



$$16\,\theta = 8\,\theta_1$$
$$\theta_1 = 2\theta$$

$$EWD = (192 \ x \ 8\theta) + (288 \ x \ 16\theta) = 6144\,\theta$$

$$IWD = M_P\theta + 2M_P(3\theta) = 7M_P\theta$$

$$IWD = EWD$$

$$7M_P\theta = 6144\,\theta$$

$$M_P = 877.71 \ kNm$$

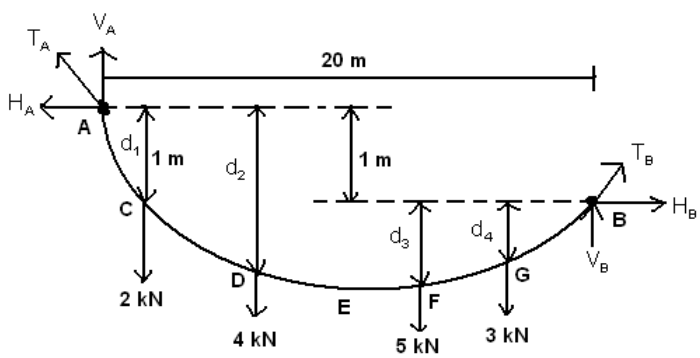
The required plastic moment of the beam section shall be $M_P = 877.71$ kNm.

UNIT 5 - SPACE AND CABLE STRUCTURES

PART - B (16 marks)

1. **A suspension cable is supported at two point "A" and "B", "A" being one metre above "B". the distance AB being 20 m. the cable is subjected to 4 loads of 2 kN, 4 kN, 5 kN and 3 kN at distances of 4 m, 8 m, 12 m and 16 m respectively from "A". Find the maximum tension in the cable, if the dip of the cable at point of application of first loads is 1 m with respect to level at A. find also the length of the cable. (AUC Apr/May 2011)**

   Solution:



Step 1: Reactions:

$$\sum V = 0$$

$$V_A + V_B = 14$$

$$\sum M @ B = 0$$

$$(V_A x\, 20) - (H\, x 1) - (2 \text{ x } 16) - (4 \text{ x } 12) - (5 \text{ x } 8) - (3 \text{ x } 4) = 0$$

$$20\, V_A - H - 132 = 0$$

$$V_A = 0.05 H + 6.6 \text{ .... } (1)$$

$$\sum H = 0$$

$$H_A = H_B$$

$$\sum M \text{ @ } C = 0$$

$$(V_A \text{ x } 4) - (H \text{ x } 1) = 0$$

$$V_A = 0.25H \quad .... (2)$$

sub. (2) in (1),

$$0.25H = 0.05H + 6.6$$

$$H = 33 kN$$

$$(1) \Rightarrow \quad V_A = 8.25 \text{ kN}$$

$$V_B = 5.75 \text{ kN}$$

Step 2 : Maximum Tension in the cable :

$$T_A = \sqrt{V_A^2 + H^2} = \sqrt{8.25^2 + 33^2} = 34.02 kN$$

$$T_B = \sqrt{V_B^2 + H^2} = \sqrt{5.75^2 + 33^2} = 33.49 kN$$

Maximum Tension in the cable, $T_{max} = 34.09 kN$.

Step 3: Length of the cable :

Here, $d_1 = 1m$

Equating moments about D to zero,

$$(8.25 \text{ x } 8) - (33 \text{ x } d_2) = 0$$

$$d_2 = 2 m$$

Equating moments about D to zero,

$$(-5.75 \text{ x } 8) + (33 \text{ x } d_3) = 0$$

$$d_3 = 1.39 m$$

Equating moments about D to zero,

$$(-5.75 \text{ x } 4) + (33 \text{ x } d_4) = 0$$

$$d_4 = 0.69 m$$

$$AC = \sqrt{4^2 + 1^2} = 4.12 \text{ m}$$

$$CD = \sqrt{4^2 + 2^2} = 4.47 \text{ m}$$

$$FG = \sqrt{4^2 + 1.39^2} = 4.23 \text{ m}$$

$$GB = \sqrt{4^2 + 0.69^2} = 4.06 \text{ m}$$

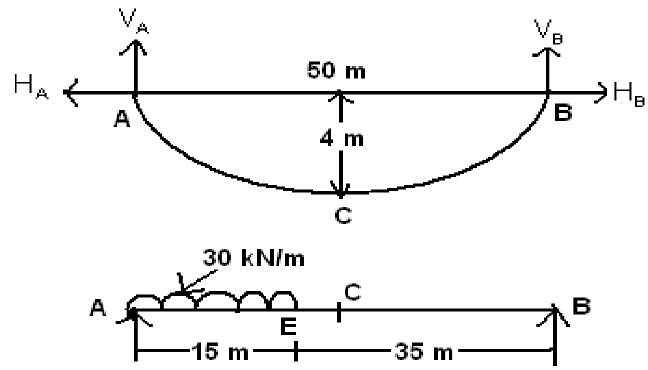
Length of the cable, $L = AC + CD + FG + BG + DF$

$$= 4.12 + 4.47 + 4.23 + 4.06 + 4$$

$$L = 20.88 m$$

2. A suspension bridge has a span 50 m with a 15 m wide runway. It is subjected to a load of 30 kN/m including self weight. The bridge is supported by a pair of cables having a central dip of 4 m. find the cross sectional area of the cable necessary if the maximum permissible stress in the cable materials is not to exceed 600 MPa. (AUC Nov/Dec 2011)

Solution:



Step 1: Reactions:

$$\sum V = 0$$

$$V_A + V_B = 450$$

$$\sum M @ A = 0$$

$$-(V_B x 50) + \left(\frac{30 \text{ x } 15^2}{2}\right) = 0$$

$$V_B = 67.5 \text{kN}$$

$$V_A = 382.5 \text{kN}$$

$$\sum H = 0$$

$$H_A = H_B$$

$$\sum M @ C = 0$$

$$(V_A \text{ x } 25) - (H \text{ x } 4) - (30 \text{ x } 15 \text{ x} (7.5+10)) = 0$$

$$H = 421.87 \text{kN}$$

Step 2: Maximum Tension in the cable:

$$T_A = \sqrt{V_A^2 + H^2} = \sqrt{382.5^2 + 421.87^2} = 569.46 \text{kN}$$

$$T_B = \sqrt{V_B^2 + H^2} = \sqrt{67.5^2 + 421.87^2} = 427.24 \text{kN}$$

Maximum Tension in the cable, $T_{max} = 569.46$ kN.

Step 3: Area:

$$T_{max} = \sigma . A$$

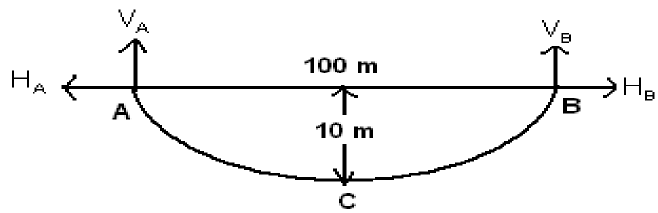
$$A = \frac{T_{max}}{\sigma} = \frac{569.46 \text{ x } 10^3}{600}$$

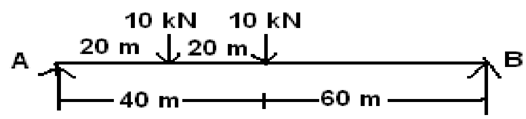
Area, $A = 949.1 \text{ mm}^2$.

3. **A three hinged stiffening girder of a suspension bridge of 100 m span subjected to two point loads 10 kN each placed at 20 m and 40 m respectively from the left hand hinge. Determine the bending moment and shear force in the girder at section 30 m from each end. Also determine the maximum tension in the cable which has a central dip of 10 m.**

(AUC May/June 2012)

Solution:





Step 1: Reactions:

$$\sum V = 0$$

$$V_A + V_B = 20$$

$$\sum M @ B = 0$$

$$(V_A \times 100) - (10 \times 80) - (10 \times 60) = 0$$

$$V_A = 14\,kN$$

$$V_B = 6\,kN$$

$$\sum H = 0$$

$$H_A = H_B$$

$$\sum M @ C = 0$$

$$(V_A \times 50) - (H \times 10) - (10 \times 30) - (10 \times 10) = 0$$

$$H = 30\,kN$$

Step 2: Shear force:

SF at 30 m from left hand hinge.

$$V_{30} = V_A - 10 - H \tan\theta$$

here,

$$\tan\theta = \frac{4d}{\ell^2}(\ell - 2x) = \frac{4 \times 10}{100^2}(100 - (2 \times 30))$$

$$\tan\theta = 0.16$$

$$V_{30} = 14 - 10 - (30 \times 0.16)$$

$$V_{30} = -0.8\,kN$$

SF at 30 m from right hand hinge.

$$V_{30} = V_B - H \tan\theta$$
$$= 6 - (30 \text{ x } 0.16)$$
$$V_{30} = 1.2 \text{ kN}$$

Step 3: Bending Moment:

BM at 30 m from left hand hinge.

$$BM_{30} = V_A \text{x} 30 - H \text{ x } y - 10 \text{ x } 10$$

here, y at 30 m from each end,

$$y = \frac{4d}{\ell^2} \text{ x } X(\ell - X^2) = \frac{4 \text{x} 10}{100^2} \text{ x } 30(100 - 30)$$
$$y = 8.4 \text{ m}$$
$$BM_{30} = (14 \text{x} 30) - (30 \text{x} 8.4) - 100 = 68 \text{ kNm.}$$

BM at 30 m from right hand hinge.

$$BM_{30} = -V_B \text{x} 30 + H \text{ x } y$$
$$= -(6 \text{ x } 30) + (30 \text{ x } 8.4)$$
$$BM_{30} = 72 \text{ kNm.}$$

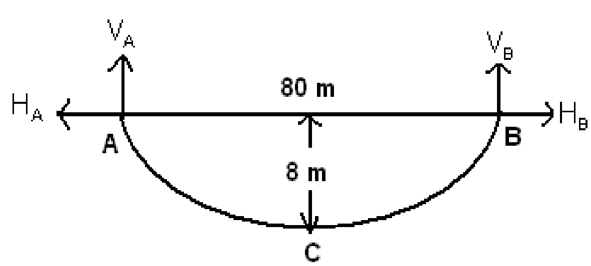
Step 4: Maximum Tension in the cable:

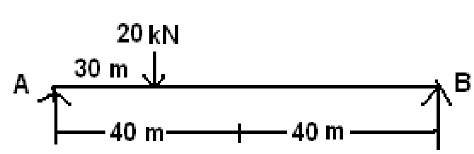
$$T_A = \sqrt{V_A^2 + H^2} = \sqrt{14^2 + 30^2} = 33.11 \text{kN}$$
$$T_B = \sqrt{V_B^2 + H^2} = \sqrt{6^2 + 30^2} = 30.59 \text{kN}$$

Maximum Tension in the cable, $T_{max} = 33.11$ kN.

4. A suspension bridge cable of span 80 m and central dip 8 m is suspended from the same level at two towers. The bridge cable is stiffened by a three hinged stiffening girder which carries a single concentrated load of 20 kN at a point of 30 m from one end. Sketch the SFD for the girder. (AUC May/June 2013)

Solution:





Step 1: Reactions:

$$\sum V = 0$$

$$V_A + V_B = 20$$

$$\sum M @ B = 0$$

$$(V_A \text{ x } 80) - (20 \text{ x } 50) = 0$$

$$V_A = 12.5 \text{ kN}$$

$$V_B = 7.5 \text{ kN}$$

$$\sum H = 0$$

$$H_A = H_B$$

$$\sum M @ C = 0$$

$$(V_A \text{ x } 40) - (20 \text{ x } 10) - (H \text{ x } 8) = 0$$

$$H = 37.5 \text{ kN}$$

Step 2: Shear force:

SF at 40 m from left hand hinge.

$$V_{40} = V_A - 20 - H \tan\theta$$

here,

$$\tan\theta = \frac{4d}{\ell^2}(\ell - 2x) = \frac{4 \text{ x } 8}{80^2}(80 - (2 \text{ x } 40))$$

$$\tan\theta = 0$$

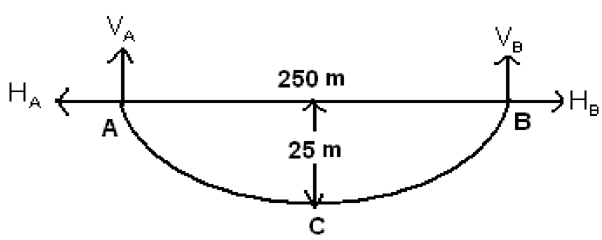
$$V_{40} = 12.5 - 20 - (37.5 \text{ x } 0)$$

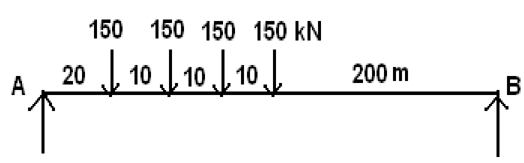
$$V_{40} = -7.5 \text{ kN}$$

5. A suspension bridge 0f 250 m span has two nos. of three hinged stiffening girders supported by cables with a central dip of 25 m. if 4 point loads of 300 kN each are placed at the centre line of the roadway at 20, 30, 40 and 50 m from left hand hinge. Find the shear force and bending moment in each girder at 62.5 m from each end. Calculate also the maximum tension in the cable.

Solution:

The load system is shared equally by the two girders and cables. Take the loads as 150 kN each.





Step 1: Reactions:

$$\sum V = 0$$

$$V_A + V_B = 600$$

$$\sum M @ B = 0$$

$$(V_A \text{ x } 250) - (150 \text{ x } 230) - (150 \text{ x } 220) - (150 \text{ x } 210) - (150 \text{ x } 200) = 0$$

$$V_A = 516 \text{ kN}$$

$$V_B = 84 \text{ kN}$$

$$\sum H = 0$$

$$H_A = H_B$$

$$\sum M @ C = 0$$

$$(V_A \text{ x } 125) - (H \text{ x } 25) - (150 \text{ x } 105) - (150 \text{ x } 95) - (150 \text{ x } 85) - (150 \text{ x } 75) = 0$$

$$H = 420 \text{ kN}$$

Step 2: Shear force:

SF at 62.5 m from left hand hinge.

$$V_{62.5} = V_A - 150 - 150 - 150 - 150 - H \tan\theta$$

here,

$$\tan\theta = \frac{4d}{\ell^2}(\ell - 2x) = \frac{4 \text{ x } 25}{250^2}(250 - (2 \text{ x } 62.5))$$

$$\tan\theta = 0.2$$

$$V_{62.5} = 516 - 150 - 150 - 150 - 150 - (420 \text{ x } 0.2)$$

$$V_{62.5} = -168 \text{ kN}$$

SF at 62.5 m from right hand hinge.

$$V_{62.5} = V_B - H \tan\theta$$

$$= 84 - (420 \text{ x } 0.2)$$

$$V_{62.5} = 0$$

Step 3: Bending Moment:

BM at 62.5 m from left hand hinge.

$$BM_{62.5} = V_A \text{ x } 62.5 - (150 \text{ x } 42.5) - (150 \text{ x } 32.5) - (150 \text{ x } 22.5) - (150 \text{ x } 12.5) - H \text{ x } y$$

here, y at 62.5 m from each end,

$$y = \frac{4d}{\ell^2} \text{ x } X(\ell - X^2) = \frac{4 \text{ x } 25}{250^2} \text{ x } 62.5(250 - 62.5)$$

$$y = 18.75 \text{ m}$$

$$BM_{62.5} = (516 \text{ x } 62.5) - (150 \text{ x } 42.5) - (150 \text{ x } 32.5) - (150 \text{ x } 22.5) - (150 \text{ x } 12.5) - (420 \text{ x } 18.75)$$

$$BM_{62.5} = 7875 \text{ kNm.}$$

BM at 62.5 m from right hand hinge.

$$BM_{62.5} = -V_B x\, 62.5 + H \text{ x } y$$
$$= -(84 \text{ x } 62.5) + (420 \text{ x } 18.75)$$
$$BM_{62.5} = 2625 \text{ kNm.}$$

Step 4 : Maximum Tension in the cable :

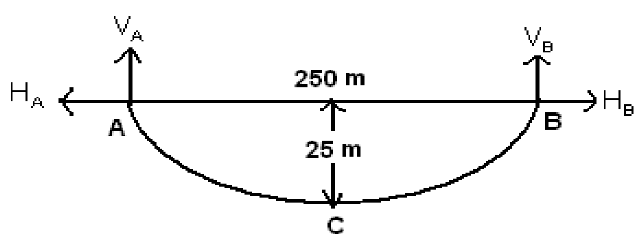
Bending moment for the cable,

$$Hd = \frac{w\ell^2}{8}$$
$$w = \frac{H \text{ x } d \text{ x } 8}{\ell^2} = \frac{420 \text{ x } 25 \text{ x } 8}{250^2} = 1.344 \text{ kN / m}$$
$$V_A = V_B = \frac{w\ell}{2} = \frac{1.344 \text{ x } 250}{2} = 168 \text{ kN}$$
$$T_{max} = \sqrt{V_A^2 + H^2} = \sqrt{168^2 + 420^2} = 452.35 \text{ kN}$$

Maximum Tension in the cable, $T_{max} = 452.35$ kN.

6. A suspension bridge is of 160 m span. The cable of the bridge has a dip of 12 m. the cable is stiffened by a three hinged girder with hinges at either end and at centre. The dead load of the girder is 15 kN/m. find the greatest positive and negative bending moments in the girder when a single concentrated load of 340 kN passes through it. Also find the maximum tension in the cable.

Solution:



Step1 : Bending Moment :

The uniformly distributed dead load will not cause any bending moment in the stiffening girder. The live load is a single concentrated moving load.

$$\text{Max. +ve BM} = 0.096\, W\ell = 0.096 \text{ x } 340 \text{ x } 160$$
$$= 5222.4 \text{ kNm.}$$

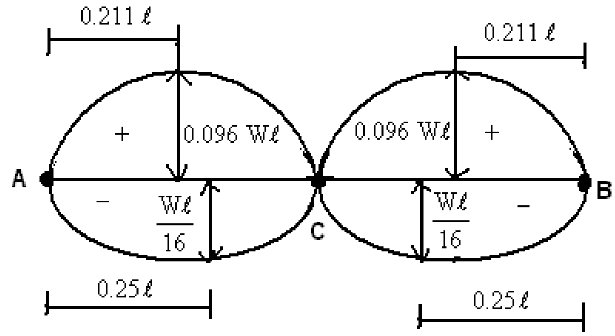
This will occur at $0.211\ell = 0.211 \text{ x } 160$

$= 33.76$ m from either end.

$$\text{Max. -ve BM} = -\frac{W\ell}{16} = -\frac{340 \text{ x } 160}{16}$$
$$= -3400 \text{ kNm.}$$

This will occur at $0.25\ell = 0.25 \text{ x } 160$

$= 40$ m from either end.



Step 2: Maximum tension in the cable:

Dead load of the girder (transmitted to the cable directly)

$$p_d = 15 \text{ kN/m}$$

Equivalent udl transmitted to the cable due to the moving concentrated load,

$$p_\ell = \frac{2 \text{ x } 340}{160} = 4.25 \text{ kN/m}$$

Total load transmitted to the cable, $p = p_d + p_\ell = 15 + 4.25 = 19.25 \text{ kN/m}$

$$\text{Vertical reaction, } V = \frac{p\ell}{2} = \frac{19.25 \text{ x } 160}{2} = 1540 \text{ kN}$$

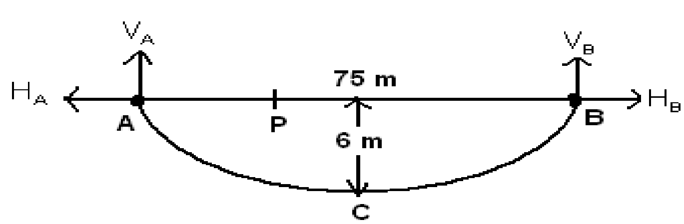
$$\text{Horizontal pull, } H = \frac{p\ell^2}{8d} = \frac{19.25 \text{ x } 160^2}{8 \text{ x } 12} = 5133.2 \text{ kN}$$

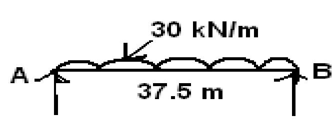
$$\text{Maximum tension, } T_{max} = \sqrt{V_A{}^2 + H^2} = \sqrt{1540^2 + 5133.3^2}$$

$$T_{max} = 5359.3 \text{ kN.}$$

7. A suspension cable of 75 m horizontal span and central dip 6 m has a stiffening girder hinged at both ends. The dead load transmitted to the cable including its own weight is 1500 kN. The girder carries a live load of 30 kN/m uniformly distributed over the left half of the span. Assuming the girder to be rigid, calculate the shear force and bending moment in the girder at 20 m from left support. Also calculate the maximum tension in the cable.

Solution:





$$\ell = 75\text{m}; \ d = 6\text{m}; \ DL = 1500 \text{ kN}; \ LL = 30 \text{ kN/m}$$

Since the girder is rigid, the live load is transmitted to the cable as an udl whatever the position of the load.

Horizontal force due to live load, $H_\ell = \frac{P\ell}{8d} = \frac{(30 \text{ x } 37.5) \text{ x } 75}{8 \text{ x } 6} = 1757.8 \text{ kN}$

Horizontal force due to dead load, $H_d = \frac{P\ell}{8d} = \frac{1500 \text{ x } 75}{8 \text{ x } 6} = 2343.8 \text{ kN}$

Total horizontal force, $H = H_\ell + H_d = 1757.8 + 2343.8 = 4101.6 \text{ kN}$

$$V_A = V_B = \frac{\text{Total load}}{2} = \frac{W_\ell + W_d}{2}$$

$$= \frac{(30 \text{ x } 37.5) + 1500}{2} = 1312.5 \text{ kN}$$

Maximum tension in the cable:

$$T_{max} = \sqrt{H^2 + V^2} = \sqrt{4101.6^2 + 1312.5^2}$$

$$T_{max} = 4306.5 \text{ kN}$$

Dip at $x = 20\text{m}$:

$$y = \frac{4d}{\ell^2} \text{ x } X(\ell - X^2) = \frac{4 \text{ x } 6}{75^2} \text{ x } 20(75 - 20) = 4.69\text{m}$$

$$\tan\theta = \frac{4d}{\ell^2}(\ell - 2x) = \frac{4 \text{ x } 6}{75^2} \text{ x}(75 - 2 \text{ x } 20) = 0.149$$

To find V_A and V_B:

$$V_A + V_B = 1125$$

Equating moments about A to zero

$$(V_B \text{ x } 75) - (30 \text{ x } 37.5 \text{ x } 18.75) = 0$$

$$V_B = 281.25 \text{ kN}$$

$$V_A = 843.75 \text{ kN}$$

Bending Moment at P:

$$BM_{20} = V_A \text{ x } 20 - H_\ell \text{ x } y - \frac{w\ell^2}{2}$$

$$= (843.75 \text{ x } 20) - (1757.8 \text{ x } 4.69) - \frac{30 \text{ x } 20^2}{2}$$

$$BM_{20} = 2630.92 \text{ kNm.}$$

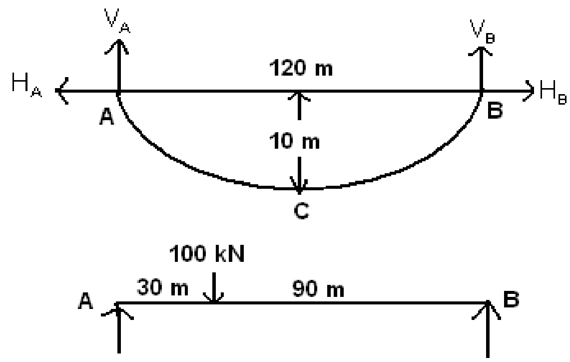
Shear force at P:

$$SF_{20} = V_A - H_\ell \text{ x } \tan\theta - w\ell = 843.75 - (1757.8 \text{ x } 0.149) - (30 \text{ x } 20)$$

$$SF_{20} = -18.16 \text{ kN.}$$

8. A suspension cable has a span of 120 m and a central dip of 10 m and is suspended from the same level at both towers. The bridge is stiffened by a stiffening girder hinged at the end supports. The girder carries a single concentrated load of 100 kN at a point 30 m from left end. Assuming equal tension in the suspension hangers. Calculate the horizontal tension in the cable and the maximum positive bending moment.

Solution:



Step 1: Reactions:

$$\sum V = 0$$

$$V_A + V_B = 100$$

$$\sum M @ A = 0$$

$$(100 \text{ x } 30) - (V_B \text{ x } 120) = 0$$

$$V_B = 25 \text{ kN}$$

$$V_A = 75 \text{ kN}$$

$$\sum H = 0$$

$$H_A = H_B$$

$$\sum M @ C = 0$$

$$-(V_B \text{ x } 60) + (H \text{ x } 10) = 0$$

$$H = 150 \text{ kN}$$

Step 2: Maximum Tension in the cable:

Bending moment for the cable,

$$w = \frac{100}{\ell} = \frac{100}{120} = 0.83 \text{ kN/m}$$

$$V_A = V_B = \frac{w\ell}{2} = \frac{0.83 \text{ x } 120}{2} = 50 \text{ kN}$$

$$T_{max} = \sqrt{V_A^2 + H^2} = \sqrt{50^2 + 150^2} = 158.1 \text{ kN}$$

Maximum Tension in the cable, $T_{max} = 158.1$ kN.

Step 3 : Maximum positive Bending Moment :

Maximum positive Bending moment will occur at under the point load.

$$BM_{30} = V_A \, x \, 30 - H \, x \, y$$

here, y at 30 m from left end,

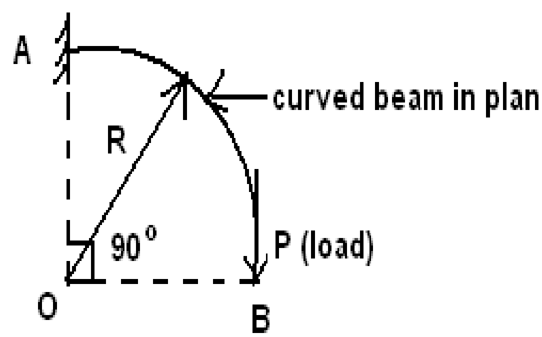
$$y = \frac{4d}{\ell^2} \, x \, X(\ell - X^2) = \frac{4 \, x \, 10}{120^2} \, x \, 30(120 - 30)$$

$$y = 7.5 \, m$$

$$BM_{30} = (75 \, x \, 30) - (150 \, x \, 7.5)$$

$$BM_{30} = 1125 \, kNm.$$

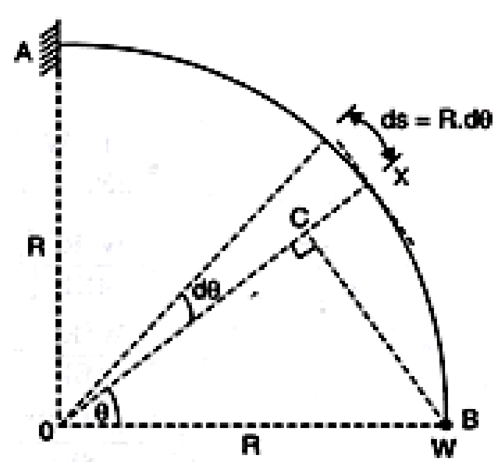
9. A quarter circular beam of radius 'R' curved in plan is fixed at A and free at B as shown in figure. It carries a vertical load P at its free end. Determine the deflection at free end and draw the bending moment and torsional moment diagrams. Assume flexural rigidity (EI) = torsional rigidity (GJ). (227) (AUC May/June 2012)



Solution:

The given cantilever is a statically determinate structure. Consider any point X on the beam at an angle θ from OB.

$$CX = R \, (1 - \cos\theta)$$



Step 1: Shear force:

SF at the section X, $F_o = W$

F_o is independent of θ and uniform throughout.

Step 2: Bending Moment:

BM at the section X, $M_\theta = -W(CB)$

$M_\theta = -W.R\sin\theta$

At $\theta = 0$, $M_B = 0$

At $\theta = \frac{\pi}{2}$, $M_A = -WR$

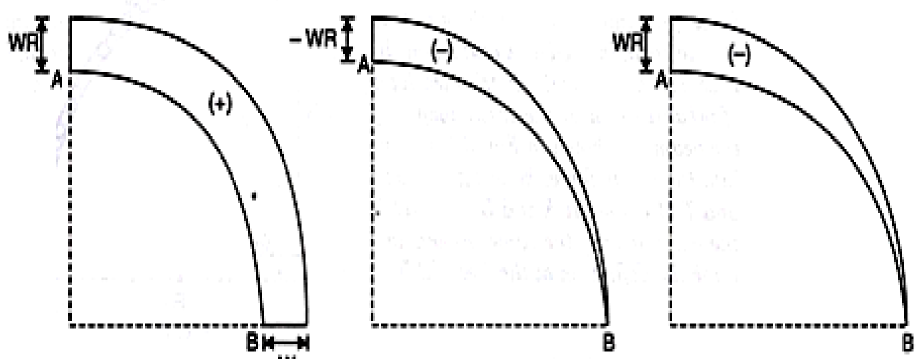
Step 3: Twisting Moment:

Twisting moment at the section X, $T_\theta = -W(CX)$

$T_\theta = -WR(1-\cos\theta)$

At $\theta = 0$, $T_B = -WR(1-\cos\theta) = 0$

At $\theta = \frac{\pi}{2}$, $T_A = -WR\left(1-\cos\frac{\pi}{2}\right) = -WR$



(a) S.F. Diagram (b) B.M. Diagram (c) T.M. Diagram

Step 4: Deflection at the free end B:

Method of strain energy is used to find the deflection at the free end B.

Strain energy, $U = \int\frac{M_\theta^2\,ds}{2\,EI} + \int\frac{T_\theta^2\,ds}{2\,GJ}$

$$= \frac{1}{2EI}\int_0^{\pi/2}(-WR\sin\theta)^2\,R\,d\theta + \frac{1}{2GJ}\int_0^{\pi/2}[-WR(1-\cos\theta)]^2\,R\,d\theta$$

$$= \frac{1}{2EI}\int_0^{\pi/2}(W^2R^2\sin^2\theta)\,R\,d\theta + \frac{1}{2GJ}\int_0^{\pi/2}[W^2R^2(1+\cos^2\theta-2\cos\theta)\,R\,d\theta$$

$$= \frac{1}{2EI}\,W^2R^3\int_0^{\pi/2}\left(\frac{1-\cos2\theta}{2}\right)d\theta + \frac{1}{2GJ}\,x\,W^2R^3\int_0^{\pi/2}\left(1+\frac{1+\cos2\theta}{2}-2\cos\theta\right)d\theta$$

$$= \frac{W^2R^3}{4EI} \int_0^{\pi/2} 1 - \cos 2\theta \;\; d\theta + \frac{W^2R^3}{4GJ} \text{ x} \int_0^{\pi/2} 2 + 1 + \cos 2\theta - 4\cos\theta \;\; d\theta$$

$$= \frac{W^2R^3}{4EI}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + \frac{W^2R^3}{4GJ} \text{ x} \left[3\theta + \frac{\sin 2\theta}{2} - 4\sin\theta\right]_0^{\pi/2}$$

$$= \frac{W^2R^3}{4EI}\left[\frac{\pi}{2}\right] + \frac{W^2R^3}{4GJ}\left[\frac{3\pi}{2} - 4\right]$$

$$U = \frac{\pi W^2R^3}{8EI} + \frac{W^2R^3}{8GJ}(3\pi - 8)$$

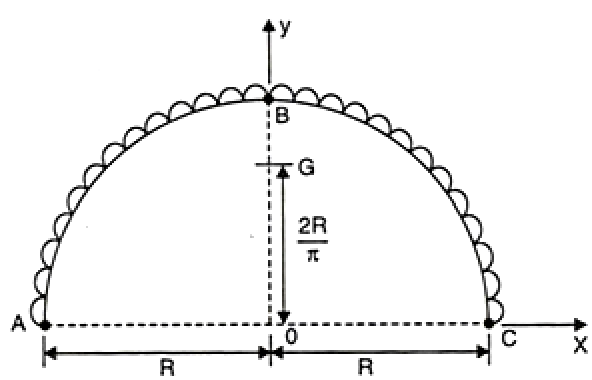
$$\delta_B = \frac{dU}{dW}$$

$$\delta_B = \frac{\pi WR^3}{4EI} + \frac{WR^3}{4GJ}(3\pi - 8)$$

10. A semicircular beam of radius 'R' in plan is subjected to udl and simply supported by three columns spaced equally. Derive the expression for bending moment and torsional moment at x be a point on the beam making an angle a' with axis passing through the base of the circle. (AUC Apr/May 2011) (AUC May/June 2013) (AUC Nov/Dec 2011)

Solution:

The curved beam given is shown in Fig. 6.10. The XX and YY axes are as shown and ZZ is the vertical axis, The unknown reactions are V_A, V_B and V_C. The end supported do not exert any moment reaction. There are three equations of static equilibrium, (*i.e.*) $\Sigma M_{xx} = 0$, $\Sigma M_{yy} = 0$ and $\Sigma F_{zz} = 0$. Hence the structure is externally determinate.



Total load on the beam = $w \times \pi R$

Vertical support reactions:

The structure, loading and support conditions are symmetric with respect to the YY axis

Hence $V_A = V_C$

Taking moments of all forces about a tangent at point B,

$$2(V_A \times R) - \pi wR\left(R - \frac{2R}{\pi}\right) = 0$$

$$V_A = \frac{1}{2R} \times \pi\, wR\left(\frac{\pi R - 2R}{\pi}\right)$$

$$= \frac{w}{2} R(\pi - 2)$$

$$V_A = \frac{(\pi - 2)}{2} wR$$

$$V_B = \text{Total load} - 2V_A$$

$$= \pi\, wR - 2x\,\frac{(\pi - 2)}{2} wR$$

$$= wR[\pi - (\pi - 2)]$$

$$V_B = 2wR$$

Bending moment and twisting moment :

Consider a section X located at an angle θ with OA

Take a segment $Rd\phi$ at an angle ϕ from x :

Bending moment, M_θ at x :

$$M_\theta = V_A \times (AN) - \int_0^\theta wR\, d\phi\, R \sin\phi$$

Using equation 6.9

$$M_\theta = \frac{(\pi - 2)}{2} wR\, R\sin\theta - \int_0^\theta wR\, d\phi\, R\sin\phi$$

$$= \frac{(\pi - 2)}{2} wR^2 \sin\theta - \int_0^\theta wR^2 \sin\phi\, d\phi$$

$$= \frac{(\pi - 2)}{2} wR^2 \sin\theta - wR^2(1 - \cos\theta)$$

$$M_\theta = wR^2\left[\frac{(\pi - 2)}{2}\sin\theta - (1 - \cos\theta)\right]$$

At $\theta = 0$,

$$M_A = wR^2\left[\frac{(\pi - 2)}{2}\sin 0 - (1 - \cos 0)\right]$$

$$M_A = 0 \text{ (hinged end)}$$

At $\theta = \frac{\pi}{2}$,

$$M_B = wR^2\left[\frac{(\pi - 2)}{2}\sin\frac{\pi}{2} - \left(1 - \cos\frac{\pi}{2}\right)\right]$$

$$= wR^2\left[\frac{(\pi - 2)}{2} \times 1 - (1 - 0)\right]$$

$$= wR^2\left[\frac{(\pi - 2)}{2} - 1\right]$$

$$= -0.429\, wR^2$$

$$M_B = -0.429\, wR^2 \qquad ...(6.12)$$

Therefore moment at B = $0.429\, wR^2$ (Hogging)

Maximum bending moment might occur between A and B (and also between B and C by symmetry

For M_θ to be maximum

$$\frac{dM_\theta}{d\theta} = 0$$

$$\frac{d}{d\theta}\left(wR^2\left[\frac{(\pi-2)}{2}\sin\theta - (1-\cos\theta)\right]\right) = 0$$

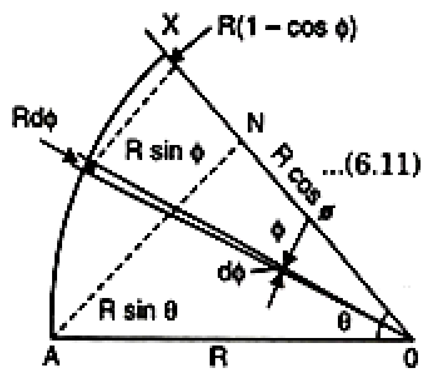
$$\frac{(\pi-2)}{2}\cos\theta - (0+\sin\theta) = 0$$

$$\frac{(\pi-2)}{2}\cos\theta - \sin\theta = 0$$

$$\frac{(\pi-2)}{2} = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{(\pi-2)}{2}$$

For M_{max}, $\theta = 29°43'$



Therefore, M_{max} $= wR^2\left[\frac{(\pi-2)}{2}\sin 29°\,43' - (1-\cos 29°\,43')\right]$

$= wR^2\,[0.2830 - 0.1315]$

$M_{max} = 0.1515\,wR^2$ sagging ...(6.14)

Maximum sagging moment = $0.1515\,wR^2$ and this will occur at $\theta = 29°43'$ from OA (OC)

Point of contraflexure :

At the point of contraflexure, $M_\theta = 0$

$$wR^2\left[\frac{(\pi-2)}{2}\sin\theta - (1-\cos\theta)\right] = 0$$

$$\frac{(\pi-2)}{2}\sin\theta - 1 + \cos = 0$$

$$\frac{(1-\cos\theta)}{\sin\theta} = \frac{(\pi-2)}{2}$$

$\theta = 59°27'$, by trial and error

Therefore the point of contraflexure occurs at $59°27'$ from OA (and from OC)

The bending moment diagram is as shown in Fig. 6.12

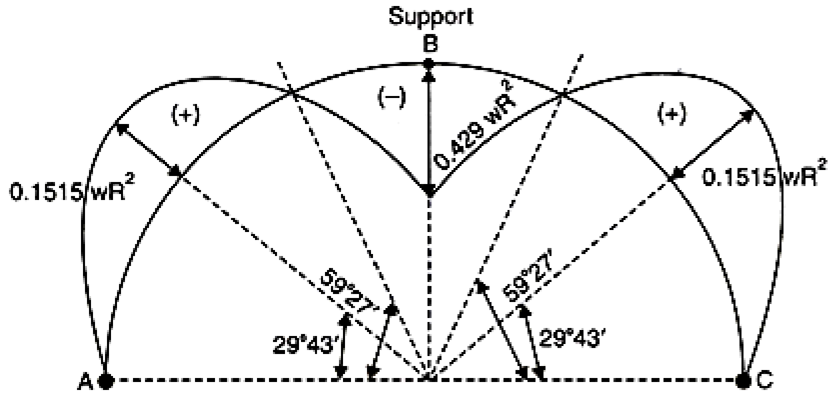


Fig. 6.12 B.M. Diagram

Twisting moment, T_θ at X :

$$T_\theta = -V_A\,(XN) + \int_0^\theta wR\, d\phi\,(1-\cos\phi)$$

$$= -\frac{(\pi-2)}{2}\, wR\,R\,(1-\cos\theta) + \int_0^\theta wR^2(1-\cos\phi)\,d\phi$$

$$= -\frac{(\pi-2)}{2}\, wR^2(1-\cos\theta) + wR^2(\theta-\sin\theta)$$

$$T_\theta = wR^2\left[-\frac{(\pi-2)}{2}(1-\cos\theta)+\theta-\sin\theta\right]$$

when $\theta = 0$, $\quad T_A = wR^2\left[-\frac{(\pi-2)}{2}(1-1)+0-0\right]$

when $\theta = \frac{\pi}{2}$, $\quad T_B = wR^2\left[-\frac{(\pi-2)}{2}(1-0)+\frac{\pi}{2}-1\right]$

$$= wR^2\left[-\frac{(\pi-2)}{2}+\frac{(\pi-2)}{2}\right] = 0$$

For maximum value of Torsional moment,

$$\frac{dT_\theta}{d\theta} = 0$$

i.e.

$$\frac{d}{d\theta}\left[wR^2\left(-\frac{(\pi-2)}{2}(1-\cos\theta)+\theta-\sin\theta\right)\right] = 0$$

$$wR^2\left[-\frac{(\pi-2)}{2}(\sin\theta)+1-\cos\theta\right] = 0$$

$$1-\cos\theta = \frac{(\pi-2)}{2}\sin\theta$$

$$\frac{1-\cos\theta}{\sin\theta} = \frac{(\pi-2)}{2}$$

$$\theta = 59°27'$$

$$T_{max} = wR^2\left[-\frac{(\pi-2)}{2}(1-\cos 59°27')+59°27'\times\frac{\pi}{180}-\sin 59°27'\right]$$

$$= wR^2\,[-0.2807+1.0376-0.8612]$$

$$= -0.1043\,wR^2$$

$$T_{max} = -0.1043\,wR^2 \qquad \text{...(6.15)}$$

The twisting moment diagram is as shown in Fig. 6.13.

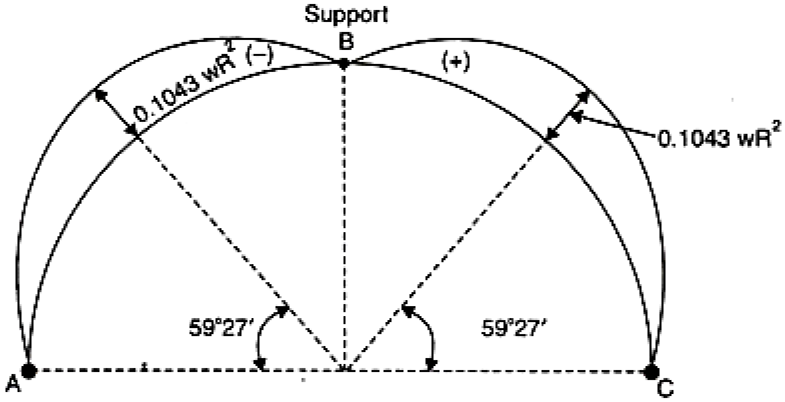


Fig. 6.13 T.M. Diagram

It is observed that the twisting moment is maximum at the section where the bending moment is zero.

Step 7 : Internal forces (P):

$$P = B\begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0.5 \\ -4.5 \\ 0.139 \\ -0.685 \end{bmatrix} = \begin{bmatrix} 3 \\ 0.861 \\ 0.139 \\ 1.185 \\ -0.685 \\ -4.5 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3 \\ 3 \\ -4 \\ 4 \\ -4.5 \\ 4.5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0.861 \\ 0.139 \\ 1.185 \\ -0.685 \\ -4.5 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 3.861 \\ -3.861 \\ 5.185 \\ -5.185 \\ 0 \end{bmatrix}$$